

Observations on entanglement entropy in massive QFT's

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ABSTRACT: We identify various universal contributions to the entanglement entropy for massive free fields. As well as the ‘area’ terms found in [1], we find other geometric contributions of the form discussed in [2]. We also compute analogous contributions for a strongly coupled field theory using the AdS/CFT correspondence. In this case, we find the results for strong and weak coupling do not agree.

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1 Introduction

Entanglement entropy has emerged as a topic of interest in a wide variety range of research areas ranging from condensed matter physics [3] to quantum gravity [4, 5]. In the context of quantum field theory (QFT), when one considers the entanglement between two regions,¹ one finds that the entanglement entropy is UV divergent because of short range correlations in the vicinity of the ‘entangling surface’ Σ separating the two regions. If the calculation is regulated with a short distance cut-off δ , the leading contributions for a QFT in D spacetime dimensions generically take the form

$$S_{\text{EE}} = \frac{c_2}{\delta^{D-2}} + \frac{c_4}{\delta^{D-4}} + \frac{c_6}{\delta^{D-6}} + \cdots, \quad (1.1)$$

where each of the coefficients c_{2k} involves an integration over the boundary Σ . For example, the leading term then yields the famous ‘area law’ result with $c_2 \propto \mathcal{A}_\Sigma$ [4]. Unfortunately the coefficients appearing in these power law divergent terms above are sensitive to the details of the UV regulator — for further discussion, see section 4.

¹These are spatial regions on a fixed Cauchy surface.

However, certain subleading contributions can reveal universal data describing the character and/or the state of the underlying QFT. A well-known example of such a contribution arises for conformal field theories (CFT's) in an even number of spacetime dimensions [6–12]. Here, in calculating the entanglement entropy, one typically finds a logarithmic contribution $\log(\delta)$ where the coefficient is some linear combination of the central charges appearing in the trace anomaly of the CFT. The precise linear combination is again determined by an integral of various geometric factors over the entangling surface. For a four-dimensional CFT, this universal contribution takes the form [9]

$$S_{\text{univ}} = \frac{\log(\delta/L)}{2\pi} \int_{\Sigma} d^2\sigma \sqrt{h} \left[a \mathcal{R}_{\Sigma} - c \left(C^{abcd} h_{ac} h_{bd} - K_a^{ib} K_b^{ia} + \frac{1}{2} K_a^{ia} K_b^{ib} \right) \right], \quad (1.2)$$

where a and c are the usual central charges which appear in the trace anomaly [13]. The various geometric factors include: h_{ab} , \mathcal{R}_{Σ} , K_{ab}^i , respectively, the induced metric, intrinsic scalar curvature and extrinsic curvature of Σ ; C_{abcd} , the (pull-back of the) Weyl curvature of the background geometry; and L , some characteristic scale in the geometry.

A similar class of universal contributions were identified in [1] for massive QFT's.² In particular, considering free massive scalar fields, the following universal contribution to the entanglement entropy was found

$$S_{\text{univ}} = \begin{cases} \gamma_D \mathcal{A}_{\Sigma} m^{D-2} \log(m\delta) & \text{for even } D, \\ \gamma_D \mathcal{A}_{\Sigma} m^{D-2} & \text{for odd } D. \end{cases} \quad (1.3)$$

where m is the mass of the scalar and \mathcal{A}_{Σ} is the area of the entangling surface. For the free scalar theory, the numerical coefficient γ_D is given by [1]

$$\gamma_{D, \text{scalar}} = \begin{cases} \frac{(-)^{D/2}}{6 (4\pi)^{\frac{D-2}{2}} \Gamma(D/2)} & \text{for even } D, \\ \frac{(-)^{\frac{D-1}{2}} \pi}{12 (4\pi)^{\frac{D-2}{2}} \Gamma(D/2)} & \text{for odd } D. \end{cases} \quad (1.4)$$

Motivated by these free field results, ref. [2] began a study of analogous terms for strongly coupled field theories using holographic techniques [8, 16]. This approach allows one to study the effect of perturbing a UV fixed point QFT by introducing general relevant operators, beyond simple mass terms. The results reveal a broad class of new universal contributions to entanglement entropy, which can be schematically

²Related results were found previously in [14] and [15].

represented as

$$S_{\text{univ}} = \gamma(D, n) \int_{\Sigma} d^{D-2}\sigma \sqrt{h} [“curvature”]^n \times \begin{cases} m^{D-2-2n} \log m\delta & \text{for even } D, \\ m^{D-2-2n} & \text{for odd } D, \end{cases} \quad (1.5)$$

where m is the mass scale appearing in the coupling of the relevant operator. The schematic geometric factor denotes a combination of both the background and extrinsic curvatures with a combined dimension $2n$. Note that $0 \leq n \leq (D-2)/2$ in these expressions. Hence in general, the entanglement entropy contains a family of universal contributions that includes the ‘area’ terms in eq. (1.3) at $n=0$ and purely geometric terms, analogous to those in eq. (1.2), at $n=(D-2)/2$. Ref. [2] focussed on universal contributions proportional to $\log(\delta)$, however, note that in general these terms (1.5) can appear for either even or odd D if the operator dimension of the relevant deformation is chosen appropriately.³ The analysis of [2] also readily extends to show the appearance of new universal terms generalizing the cut-off independent terms in eq. (1.3) for odd D . An important distinction is, however, that generally these cut-off independent terms depend on the underlying state of the boundary theory while the logarithmic terms are state independent.

One of our objectives in the following is to extend the analysis of [1] to reveal the new curvature terms appearing in eq. (1.5). This is easily accomplished by performing the free field calculations on a curved background. In particular, ref. [1] work with a ‘waveguide’ geometry, $\mathbb{R}^2 \times \mathbb{I}^{D-2}$, where \mathbb{I} is a finite interval with either Dirichlet or Neumann boundary conditions imposed at the endpoints. In section 2, we perform analogous calculations on a ‘spherical waveguide’, $\mathbb{R}^2 \times \mathbb{S}^{D-2}$. Our analysis also provides a number of other interesting extensions of that in [1]. As well as considering the entanglement entropy, we also calculate the Rényi entropy, which is another useful measure of entanglement [17, 18]. In section 2.1, we consider a free massive scalar field but we also include the possibility of the curvature coupling $\frac{1}{2}\xi R\phi^2$. In section 2.2, we extend the analysis to consider a free massive fermion. For the fermions, we find that the coefficients of the universal ‘area’ terms in eq. (1.3) become

$$\gamma_{D, \text{fermion}} = 2^{\lfloor \frac{D}{2} \rfloor - 1} \gamma_{D, \text{scalar}} = \begin{cases} \frac{(-)^{D/2}}{6(2\pi)^{\frac{D-2}{2}} \Gamma(D/2)} & \text{for even } D, \\ \frac{(-)^{\frac{D-1}{2}} \pi}{12\sqrt{2}(2\pi)^{\frac{D-2}{2}} \Gamma(D/2)} & \text{for odd } D, \end{cases} \quad (1.6)$$

³For example, choosing an operator with $\Delta = (D+2)/2$, the corresponding coupling would take the form $gm^{(D-2)/2}$ where g is a dimensionless coefficient. In this case, an ‘area’ term with $n=0$ appears for all $D \geq 3$ with $\gamma \propto g^2$.

as was noted previously in [19]. Above, $\lfloor D/2 \rfloor$ denotes the integer part of $D/2$. Our analysis in these sections also allows us to identify a particular curvature contribution (1.5) to the entanglement entropy as

$$S_{\text{univ}} = \hat{\gamma}_D \int_{\Sigma} d^{D-2} \sigma \sqrt{h} \mathcal{R}(h) \times \begin{cases} m^{D-4} \log(m\delta) & \text{for even } D \geq 4, \\ m^{D-4} & \text{for odd } D \geq 5. \end{cases} \quad (1.7)$$

where $\mathcal{R}(h)$ is the Ricci scalar of the metric induced on the entangling surface. For a free scalar with a curvature coupling, the new numerical coefficient $\hat{\gamma}_D$ is given by

$$\hat{\gamma}_{D, \text{scalar}} = \frac{D-2}{2} \left(\xi - \frac{1}{6} \right) \gamma_{D, \text{scalar}} \quad (1.8)$$

where $\gamma_{D, \text{scalar}}$ is precisely the coefficient appearing in eq. (1.4). For a massive free fermion, this coefficient can be expressed as

$$\hat{\gamma}_{D, \text{fermion}} = \frac{D-2}{24} \gamma_{D, \text{fermion}} \quad (1.9)$$

where $\gamma_{D, \text{fermion}}$ is given by eq. (1.6).

In section 3, we turn to a holographic calculation of entanglement entropy for the strongly coupled $\mathcal{N} = 2^*$ gauge theory, a massive deformation of the celebrated $\mathcal{N} = 4$ super-Yang-Mills theory [20, 21]. While similar calculations already appear in [2], the details of the boundary mass terms and their translation to the dual gravity theory are precisely understood in this well-studied framework [22–24]. Hence, we are able to compare the ‘area’ contribution (1.3) at strong coupling from holography to the weak coupling results, which combine eqs. (1.4) and (1.6). Our final conclusion is that the strong coupling result does *not* match the corresponding contribution (1.3) found at weak coupling!

We conclude the paper with a brief discussion of our results and future directions in section 4. This is followed by two appendices presenting various technical results: Appendix A explicitly demonstrates the validity of the separation of variables (2.11) used in section 2.2 for spin- $\frac{1}{2}$ fields. Finally, Appendix B repeats the calculations for section 2 for a ‘hyperbolic’ waveguide $\mathbb{R}^2 \times \mathbb{H}^{D-2}$.

2 Rényi entropy on a spherical waveguide

In general to define the entanglement or Rényi entropies of some quantum system, we begin by dividing the degrees of freedom into two subsets, A and \bar{A} , and calculate the reduced density matrix $\rho_A = \text{Tr}_{\bar{A}}(\rho)$ where ρ describes the global state of the system. In a QFT, this is typically realized by beginning with a Cauchy surface in

a fixed background and introducing of an ‘entangling surface’ Σ , which divides this surface in two separate regions, again, denoted A and \bar{A} . The reduced density matrix ρ_A is then given by integrating over all field configurations in the region \bar{A} . Given this density matrix, the entanglement entropy is then defined by the standard von Neumann formula

$$S_{\text{EE}} = -\text{Tr}(\rho_A \log \rho_A) . \quad (2.1)$$

while the Rényi entropy is given by [17]

$$S_\alpha = \frac{1}{1-\alpha} \log \text{Tr}(\rho_A^\alpha) . \quad (2.2)$$

The latter is usually evaluated for (positive) integer values of α , in which case, eq. (2.2) involves a somewhat simpler calculation since it does not require evaluating the logarithm of ρ_A appearing in eq. (2.1). Further, if the result of S_α can be continued to real values of α , (as will be possible in the following,) the entanglement entropy can be recovered as the limit: $S_{\text{EE}} = \lim_{\alpha \rightarrow 1} S_\alpha$.

The trace required in eq. (2.2) has a standard path integral representation, *e.g.*, [25, 26],

$$\text{Tr}(\rho_A^\alpha) = Z_\alpha / (Z_1)^\alpha , \quad (2.3)$$

where α is again assumed to take integer values for the moment. Implicitly, the first step here is to Wick rotate the background geometry to Euclidean time $t_E = it$. Then Z_α is the partition function of the QFT evaluated on an α -fold cover of the Euclidean background where a cut is introduced throughout region A on the Cauchy surface, which we denote $t_E = 0$. At the cut, the fields on the k ’th sheet are joined to the fields on the $(k+1)$ ’th sheet when approaching from $t_E \rightarrow 0^-$ and to those on the $(k-1)$ ’th sheet when approaching from $t_E \rightarrow 0^+$. Hence Z_α is the partition function evaluated on a singular covering geometry with an angular excess of $2\pi(\alpha - 1)$ at the entangling surface Σ . The factors of the standard partition function Z_1 appear in the denominator of eq. (2.3) to ensure that the density matrix is properly normalized with $\text{Tr}(\rho_A) = 1$. Given these partition functions, the Rényi entropy (2.2) becomes

$$S_\alpha = \frac{\log Z_\alpha - \alpha \log Z_1}{1 - \alpha} . \quad (2.4)$$

Before proceeding with our explicit calculations, let us introduce the following shorthand notation for simplicity

$$d \equiv D - 2 . \quad (2.5)$$

Now following [1], our calculations here will focus on free fields in a waveguide geometry. In particular, throughout this section, the background geometry will take

the form $\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^d$, where \mathbb{R}^2 is covered by Cartesian coordinates, t_E and x . As in the above discussion, the Cauchy surface is selected by simply setting $t_E = 0$ in which case the resulting spatial slice has the waveguide geometry $\mathbb{R} \times \mathbb{S}^d$. This slice is then divided into two halves by choosing the entangling surface to be the sphere at $x = 0$. Implicitly we assume that the QFT on \mathcal{M} is in its ground state and as described above, we are considering the reduced density matrix on the region $A = \{t_E = 0, x > 0\}$ resulting after integrating out the field degrees of freedom in \bar{A} .

To calculate the Rényi entropy as described above, we need to evaluate the partition function on \mathcal{M}_α , the α -fold cover of \mathcal{M} . At this point, we note that the background geometry has a rotational symmetry in the plane around the point $(t_E, x) = (0, 0)$, which serves as our entangling surface. Hence in constructing \mathcal{M}_α , we are simply replacing the \mathbb{R}^2 component of \mathcal{M} by a two-dimensional cone C_α with an angular excess of $2\pi(\alpha - 1)$ at the origin, *i.e.*, $\mathcal{M}_\alpha = C_\alpha \times \mathbb{S}^d$. For later calculations, we explicitly write the metric on C_α as⁴

$$ds^2 = \alpha^{-2}(dr)^2 + r^2(d\theta)^2, \quad (2.6)$$

where r and θ possessing the full radial and angular range $0 \leq r \leq \infty$, $0 \leq \theta \leq 2\pi$. Now this geometry has no distinguishing features which prefer integer values of α (apart from $\alpha = 1$) and so from this point forward, we allow α to take any (positive) real value. That is, we are analytically continuing α already in the covering geometry [27] rather than first evaluating Z_α for integer α and then analytically continuing. The rotational symmetry in the transverse space around the entangling surface is an essential ingredient for this geometric approach.⁵

One feature, which distinguishes our background here from that in [1], is that the cross-section of the waveguide \mathbb{S}^d is curved and hence our calculations of the entanglement entropy below can reveal new universal contributions of the form given in eq. (1.5). After a few more preliminary remarks, we will consider a massive free scalar field in section 2.1, with the action

$$S(\phi) = \int_{\mathcal{M}_\alpha} d^D x \sqrt{g} \frac{1}{2} \left((\nabla \phi)^2 + m^2 \phi^2 + \xi \mathcal{R} \phi^2 \right), \quad (2.7)$$

where \mathcal{R} is the Ricci scalar of the background geometry. Hence we have included a non-minimal coupling to the curvature of the background here. Of course, with $m = 0$ and $\xi = \frac{D-2}{4(D-1)}$, we have a conformal scalar field theory. In section 2.2, we also consider a

⁴After changing variables $r \rightarrow r/\alpha$, $\theta \rightarrow \alpha\theta$, one can readily conclude that the angular excess is given by $2\pi(\alpha - 1)$. However, we use the present coordinates (2.6) in the following, since the angular momentum operator takes the standard form in this representation of the cone.

⁵See [10] for further discussion.

massive free fermion field (with minimal coupling to the background geometry), which becomes a conformal theory if $m = 0$.

For either of the above classes of theories, the partition function is Gaussian and can be exactly evaluated using the heat kernel approach, *e.g.*, [28]

$$\log Z_\alpha^{(s)} = \frac{e^{i2\pi s}}{2} \int_{\delta^2}^\infty \frac{dt}{t} \text{Tr} K_{\mathcal{M}_\alpha}^{(s)} e^{-tm_s^2}, \quad (2.8)$$

where $K_{\mathcal{M}_\alpha}^{(s)}(t, x, y)$ with $x, y \in \mathcal{M}_\alpha$ is the heat kernel of the corresponding *massless* wave operator on \mathcal{M}_α . The trace of the heat kernel involves taking the limit of coincident points, *i.e.*, $y \rightarrow x$, and integrating over the remaining position x . Of course, a trace is also taken over the spinor indices in the case of the spin- $\frac{1}{2}$ field — see below. In the above expression and throughout the following, we use $s = 0$ or $\frac{1}{2}$ to indicate the scalar or fermion cases, respectively. Further, δ is a short-distance scale introduced to regulate any potential UV divergences (as discussed in the introduction). Finally m_s denotes the ‘effective’ mass of the field under study. For the fermion, we have simply $m_{s=\frac{1}{2}} = m$, however, given the non-minimal coupling of the scalar in eq. (2.7), we have

$$m_{s=0}^2 = m^2 + \xi \frac{d(d-1)}{R^2}, \quad (2.9)$$

where the second term comes from the curvature of the \mathbb{S}^d , *i.e.*, $\mathcal{R}(\mathbb{S}^d) = d(d-1)/R^2$ for a sphere of radius R .

Notice that the curvature of the full background geometry contains a singularity at the tip of the cone, *e.g.*,

$$\mathcal{R}(\mathcal{M}_\alpha) = \mathcal{R}(\mathbb{S}^d) + \mathcal{R}(C_\alpha) = \frac{d(d-1)}{R^2} + 4\pi(1-\alpha)\delta^{(2)}(\vec{r}) + \dots \quad (2.10)$$

where $\vec{r} \in C_\alpha$ and the second term corresponds to the leading contribution from $\mathcal{R}(C_\alpha)$ in an expansion near $\alpha \simeq 1$. To treat this singularity in a well defined way, we delete the point at the tip of C_α and work on the space $(C_\alpha - \{0\}) \times \mathbb{S}^d$. Of course, this means that appropriate boundary conditions must be imposed at the tip to make the wave operator self-adjoint. This then becomes the requirement that we must use only non-singular eigenfunctions when the heat kernel is constructed [29]. To simplify the notation we use the notation $\mathcal{M}_\alpha = C_\alpha \times \mathbb{S}^d$ to denote the punctured manifold in what follows.

The wave operators are separable on the product manifold $C_\alpha \times \mathbb{S}^d$ and hence the heat kernel on \mathcal{M}_α can be expressed as the product of the two individual heat kernels on C_α and \mathbb{S}^d , *i.e.*,

$$K_{\mathcal{M}_\alpha}^{(s)} = K_{C_\alpha}^{(s)} K_{\mathbb{S}^d}^{(s)}, \quad (2.11)$$

where for brevity we have suppressed the arguments of the heat kernels here. Note that while this separation of variables is obvious in the case of the spin-0 field, it is less evident in the case of the spin- $\frac{1}{2}$ field due to spinor structure of the heat kernel. Hence we show that separation of variables indeed holds in the latter case in Appendix A. Given eq. (2.11), one can write

$$\text{Tr } K_{\mathcal{M}_\alpha}^{(s)} = \text{Tr } K_{C_\alpha}^{(s)} \text{Tr } K_{\mathbb{S}^d}^{(s)}, \quad (2.12)$$

where each trace on the right-hand side involves an integration over the corresponding component of the product manifold. In the case of spin- $\frac{1}{2}$ field, there is also trace over spinor indices. Using the conventions established in Appendix A, we can regard the two traces on the right as also including a separate trace over the spinor spaces of the two component manifolds, C_α and \mathbb{S}^d .

In eq. (2.8), the possible UV divergences at $t \rightarrow 0$ in the partition function were regulated by introducing the short-distance cut-off δ . However, we would also like to introduce a ζ -function regularization [28] here since it readily allows us to identify the universal contributions to the Rényi entropy for general values of d . This approach will be applied in calculating to the Rényi entropy of the scalar field on $C_\alpha \times \mathbb{S}^d$ in the next subsection. However, we also apply this regularization to produce general results for both scalars and fermions on the hyperbolic waveguide $C_\alpha \times \mathbb{H}^d$ in Appendix B.

In the ζ -function approach, the partition function is regulated by shifting the power of t in eq. (2.8)

$$\log Z_\alpha^{(s)} = \frac{e^{i2\pi s}}{2} \delta^{-2z} \int_0^\infty \frac{dt}{t^{1-z}} \text{Tr } K_{\mathcal{M}_\alpha}^{(s)} e^{-tm_s^2}, \quad (2.13)$$

where now δ appears to keep the whole expression dimensionless. Of course, after carrying out the integral over t , the regulator must be removed by taking the limit $z \rightarrow 0$ and suitably renormalizing the parameters of the theory to eliminate possible divergences in z .

Now, as shown in [29, 30], the trace of the heat kernel on a cone depends on α only, therefore substituting eq. (2.12) into eq. (2.13) yields

$$\log Z_\alpha^{(s)} = \frac{e^{i2\pi s}}{2} \delta^{-2z} \text{Tr } K_{C_\alpha}^{(s)} \Gamma(z) \zeta_{\mathbb{S}^d}^{(s)}(z), \quad (2.14)$$

where the ζ -function is defined as follows

$$\zeta_{\mathbb{S}^d}^{(s)}(z) = \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \text{Tr } K_{\mathbb{S}^d}^{(s)} e^{-tm_s^2}. \quad (2.15)$$

Expanding eq. (2.14) in the vicinity of $z = 0$ then yields

$$\log Z_\alpha^{(s)} = \frac{e^{i2\pi s}}{2} \text{Tr } K_{C_\alpha}^{(s)} \left[\frac{\zeta_{\mathbb{S}^d}^{(s)}(0)}{z} + \frac{d\zeta_{\mathbb{S}^d}^{(s)}}{dz} \Big|_{z=0} - \zeta_{\mathbb{S}^d}^{(s)}(0) \log \delta^2 + \mathcal{O}(z) \right], \quad (2.16)$$

where we rescaled $\delta^2 \rightarrow e^{-\gamma}\delta^2$ to absorb a term proportional to the Euler constant γ . The pole term in the above expression must be removed by suitably renormalizing the field theory parameters.⁶ The remaining contributions are precisely those which determine universal contributions to the Rényi entropy

$$\log Z_\alpha^{(s)} = \frac{e^{i2\pi s}}{2} \text{Tr} K_{C_\alpha}^{(s)} \left[\left. \frac{d\zeta_{\mathbb{S}^d}^{(s)}}{dz} \right|_{z=0} - \zeta_{\mathbb{S}^d}^{(s)}(0) \log \delta^2 \right] . \quad (2.17)$$

This expression is readily evaluated using results available in the literature [31, 32]. The ζ -function on the hyperbolic space \mathbb{H}^d was computed for the spin-0 case in [31] and for spin- $\frac{1}{2}$ case in [32]. For the scalars, the desired ζ -functions on \mathbb{S}^d are then easily obtained from the hyperbolic ones using a formula given in [31] — see eq. (2.25).

2.1 Rényi entropy for a massive scalar

We begin here by evaluating the partition function Z_α for the massive free scalar field theory described by the action (2.7). In this case, the heat kernel in eq. (2.8) corresponds to the D -dimensional scalar Laplacian on \mathcal{M}_α . Separation of the variables leads to eq. (2.12) and further the heat kernel on C_α is given by [33]

$$\text{Tr} K_{C_\alpha}^{(0)}(t) = \frac{1}{12\alpha} (1 - \alpha^2) + \alpha \text{Tr} K_{\mathbb{R}^2}^{(0)}(t) . \quad (2.18)$$

On the other hand, the scalar heat kernel on \mathbb{S}^d satisfies

$$\begin{aligned} (-\partial_t + \Delta^{(d)}) K_{\mathbb{S}^d}^{(0)}(t, x, y) &= 0 , & x, y \in \mathbb{S}^d \\ K_{\mathbb{S}^d}^{(0)}(0, x, y) &= \delta(x, y) , \end{aligned} \quad (2.19)$$

where $\Delta^{(d)}$ is the scalar Laplacian on \mathbb{S}^d . Of course, due to the rotational symmetry the heat kernel only depends on the arc-length between the two points on the sphere. Therefore for simplicity, we place one of the points at the north pole of the sphere. With this choice, the heat kernel becomes a function of only the azimuthal angle θ and we may replace $\Delta^{(d)}$ by its radial part.

To solve the resulting equation, we follow the prescription described in [32]. That is, we consider the intertwining operator, $\mathcal{O} = -(2\pi \sin \theta)^{-1} \partial_\theta$, which relates the Laplacian on spheres of different dimensions, *i.e.*,

$$\Delta_\theta^{(d)} \mathcal{O} = \mathcal{O} \left(\Delta_\theta^{(d-2)} + \frac{d-2}{R^2} \right) , \quad \Delta_\theta^{(d)} = \partial_\theta^2 + (d-1) \cot \theta \partial_\theta . \quad (2.20)$$

⁶Appearance of the logarithmic term in eqs. (2.16) and (2.17) might seem misleading. Indeed, such terms are expected in the case of even d only. However, as follows from eq. (2.25) and eq. (B.4) for $s = 0$ or eq. (B.25) for $s = 1/2$, $\zeta_{\mathbb{S}^d}^{(s)}(0) = 0$ for odd d and so the $\log \delta$ term vanishes as expected.

Here R is the radius of both \mathbb{S}^d and \mathbb{S}^{d-2} . The overall constant factor in \mathcal{O} is chosen such that \mathcal{O} relates the delta functions on the two spheres. Hence, for even and odd dimensions, we have

$$K_{\mathbb{S}^{2n+1}}^{(0)}(t, \theta) = e^{n^2 t / R^2} \left(-\frac{1}{2\pi R^2 \sin \theta} \partial_\theta \right)^n K_{\mathbb{S}^1}^{(0)}(t, \theta) , \quad (2.21)$$

$$K_{\mathbb{S}^{2n+2}}^{(0)}(t, \theta) = e^{n(n+1)t/R^2} \left(-\frac{1}{2\pi R^2 \sin \theta} \partial_\theta \right)^n K_{\mathbb{S}^2}^{(0)}(t, \theta) . \quad (2.22)$$

For $d = 1$, $K_{\mathbb{S}^1}^{(0)}(t)$ can be readily evaluated using the method of images. It is given by an infinite sum of the scalar heat kernels on \mathbb{R} , which are shifted by integer multiples of 2π with respect to each other to maintain periodic boundary conditions for the scalar field on a circle, namely

$$K_{\mathbb{S}^1}^{(0)}(t, \theta) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{R^2(\theta+2\pi n)^2}{4t}} . \quad (2.23)$$

For $d = 2$, the heat kernel can be constructed using spherical harmonics $Y_{lm}(\theta, \phi)$, which correspond to the orthonormal eigenfunctions of the Laplacian on a unit two-sphere. If one of the points is taken to the north pole, the result then simplifies to the following sum

$$K_{\mathbb{S}^2}^{(0)}(t, \theta) = \frac{1}{4\pi R^2} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) e^{-\frac{l(l+1)}{R^2} t} . \quad (2.24)$$

We will also use the ζ -function formulae given in eq. (2.17). As shown in [31], ζ -functions on \mathbb{S}^d and \mathbb{H}^d for $d \geq 3$ are related and can be obtained from each other by means of complex contours. The final result in the case of spin-0 case reads

$$\begin{aligned} \zeta_{\mathbb{S}^d}^{(0)}(z) = e^{i\pi(z-d/2)} \text{Vol}(\mathbb{S}^d) & \left[\frac{\zeta_{\mathbb{H}^d}^{(0)}(z)}{\text{Vol}(\mathbb{H}^d)} - iR^{2z-d} e^{-i\pi z} \frac{2^{d-2} \Gamma(d/2)}{\pi^{d/2}} \sin(\pi z) \right. \\ & \left. \times \int_0^\infty \frac{f(ib-y)dy}{[1 + (-1)^d e^{2\pi(y-ib)}][(y-ib)^2 + b^2]^z} \right] \quad \text{with } \text{Re}(z) < 1 , \end{aligned} \quad (2.25)$$

where $\zeta_{\mathbb{H}^d}^{(0)}$ denotes the scalar ζ -function on \mathbb{H}^d , which is given by eqs. (B.4) and (B.6). Following the notation of [31], we also have the following definitions:

$$\begin{aligned} b^2 & \equiv -R^2 m_0^2 + \frac{(d-1)^2}{4} = -R^2 m^2 - \xi d(d-1) + \frac{(d-1)^2}{4} , \\ f(y) & \equiv y \frac{y^2 + (d-3)^2/4}{4^{d-2} \Gamma(d/2)^2} \prod_{k=-(d-5)/2}^{(d-5)/2} (y + ik) , \end{aligned} \quad (2.26)$$

where we have used eq. (2.9) in the second expression for b .⁷ Further for $d = 3$ and 4 , the product appearing as the last factor in $f(y)$ should be omitted.

Even though eq. (2.25) for ζ -function is only valid for $\text{Re}(z) < 1$, this will be sufficient to compute the Rényi entropy in the present context since according to eqs. (2.4) and (2.17), we only need to know ζ -function in the vicinity of $z = 0$. Note, however, that general expression which is valid for all values of z can be found in [31].

The ζ -function approach is similar to dimensional regularization in that no power law divergences will appear with this method. Hence the leading contributions to the entanglement entropy of the form given in eq. (1.1) are somewhat obscure in this framework.⁸ In contrast, evaluating the entanglement (and Rényi) entropy using eq. (2.8), where δ directly cuts off the UV end of the t integral, produces explicit power law divergences as appear in eq. (1.1). We illustrate these differences by applying both approaches in the examples below. Since the form of the heat kernels and ζ -functions is different in even and odd dimensions, we consider these cases separately.

Odd dimensions

We start to implement eq. (2.8) in the special cases $d = 1$ and 3 to illustrate how the divergent ‘area law’ and subleading terms emerge, as well as the universal ‘area’ terms (1.3). We then consider the ζ -function method (2.17) to evaluate the finite contributions to the Rényi entropy for general odd d .

$d = 1$ ($D = 3$):

In this case one has to substitute eqs. (2.18) and (2.23) into eqs. (2.4) and (2.8)

$$\begin{aligned} S_\alpha^{(0)} &= \frac{1+\alpha}{24\alpha} \pi^{1/2} R \sum_{n=-\infty}^{\infty} \int_{\delta^2}^{\infty} \frac{dt}{t^{3/2}} e^{-tm^2 - \frac{(\pi R n)^2}{t}} \\ &= \frac{1+\alpha}{12\alpha} \left(\frac{\text{Vol}(\mathbb{S}^1)}{\sqrt{4\pi}\delta} - \log[2 \sinh(\pi m R)] \right). \end{aligned} \quad (2.27)$$

Note that no ξ dependence appears in the expressions above because the curvature scalar vanishes on \mathbb{S}^1 , *i.e.*, $m_0 = m$ for $d = 1$ in eq. (2.9). The divergent term is, of course, the expected ‘area law’ contribution. It originates from the $n = 0$ summand in eq. (2.23) and thus is independent of the cross-section of the waveguide geometry, *e.g.*, the same term arises in eq. (B.11) for a hyperbolic waveguide.

⁷This notation is not ideal in the following where we focus on the limit $mR \gg 1$. Hence let us resolve the possible ambiguity by adding that $b \simeq i m R$ in this limit.

⁸Of course, this also illustrates the sensitivity of these contributions to the details of the UV regulator.

In examining the finite contribution above, we first note that it does not have the simple form expected in eq. (1.3). However, we note that the full calculations in [1] resulted in a similarly complicated expression and the simple universal term only emerged in the large mass limit. Hence we consider the finite term above in the limit $mR \gg 1$,

$$S_{\alpha, \text{finite}}^{(0)} = -\frac{1+\alpha}{24\alpha} \left(2\pi R m - 2e^{-2\pi m R} - e^{-4\pi m R} + \dots \right). \quad (2.28)$$

Here we see that the leading term has precisely the form expected in eq. (1.3) with $\mathcal{A}_\Sigma = 2\pi R$ and $D = 3$. We emphasize that above expression describes the Rényi entropy and can be evaluated for any α . The entanglement entropy is recovered by substituting $\alpha = 1$, in which case the pre-factor becomes $-1/12$ precisely matching the coefficient given in eq. (1.4) for $D = 3$. Here we also find higher order terms suppressed by exponentials $\exp(-2\pi n m R)$. Similar exponential terms were found in [1] but the precise numerical prefactors do not agree for the waveguide geometry studied there and for the present cylindrical waveguide. Of course, these terms only become important when the Compton wavelength of the scalar is comparable to the size of the cross-section of the waveguide. Hence it seems these contributions are probing the topology of the background geometry. In fact, in eq. (B.12), we find that no such exponentials arise when the cross-section is $\mathbb{H}^1 \simeq \mathbb{R}^1$.

$d = 3$ ($D = 5$):

Taking the limit $\theta \rightarrow 0$ in eq. (2.21), yields

$$K_{\mathbb{S}^3}^{(0)}(t, 0) = \frac{e^{t/R^2}}{(4\pi t)^{3/2}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2 R^2 n^2}{t}} \left(1 - 2 \frac{\pi^2 R^2 n^2}{t} \right). \quad (2.29)$$

Combining this expression with eq. (2.18) to form $K_{\mathcal{M}_\alpha}^{(0)}$ and in turn, substituting the result into eqs. (2.4) and (2.8) yields

$$\begin{aligned} S_\alpha^{(0)} &= \frac{1+\alpha}{24\alpha} \text{Vol}(\mathbb{S}^3) \int_{\delta^2}^{\infty} \frac{dt}{t} e^{-t(m^2 + 6\xi/R^2)} K_{\mathbb{S}^3}^{(0)}(t, 0) \\ &= \frac{1+\alpha}{12\alpha} \frac{\text{Vol}(\mathbb{S}^3)}{(4\pi)^{3/2}} \left[\frac{1}{3\delta^3} - \left(m^2 + \frac{6\xi - 1}{R^2} \right) \frac{1}{\delta} \right] + \frac{1+\alpha}{96\pi^2\alpha} g^{(0)} \left(2\pi \sqrt{(mR)^2 + 6\xi - 1} \right), \end{aligned} \quad (2.30)$$

with $\text{Vol}(\mathbb{S}^3) = 2\pi^2 R^3$ and

$$g^{(0)}(x) = \frac{1}{6} x^3 + x^2 \log(1 - e^{-x}) - 2 \text{Li}_3(e^{-x}) - 2x \text{Li}_2(e^{-x}), \quad (2.31)$$

where $\text{Li}_3(z)$ and $\text{Li}_2(z)$ are the standard polylogarithms. In eq. (2.30), we see the expected area law contribution proportional to $\text{Vol}(\mathbb{S}^3)/\delta^3$, as well as a subleading

divergences proportional to $1/\delta$. Focusing on the finite contribution in the limit $mR \gg 1$, we find

$$S_{\alpha,finite}^{(0)} = \frac{1+\alpha}{144\pi\alpha} \mathcal{A}_\Sigma \left(m^3 + \frac{3(6\xi-1)}{2} \frac{m}{R^2} + \frac{3(6\xi-1)^2}{8} \frac{1}{m R^4} + \dots \right. \\ \left. - \frac{3}{\pi} \frac{m^2}{R} e^{-2\pi m R} + \dots \right) \quad (2.32)$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{S}^3) = 2\pi^2 R^3$ is the area of the entangling surface. The leading term in this expansion has the form expected from eq. (1.3) and setting $\alpha = 1$ to recover the entanglement entropy, the prefactor becomes $1/(72\pi)$ which precisely matches the coefficient given in eq. (1.4) with $D = 5$. There are two classes of subleading terms: First there is an expansion in powers of $1/(mR)^2$, which produces terms where the prefactor becomes $\mathcal{A}_\Sigma m^3/(mR)^{2n}$. These contributions have precisely the form expected for the curvature terms described in eq. (1.5). Second, there are contributions with exponential factors $\exp(-2\pi n m R)$, similar to those found with $d = 1$. As explained above, it appears that these contributions probe the topology of the waveguide geometry. Note that, as illustrated in eq. (2.32), some of these exponential contributions contain odd⁹ powers of $1/R$, which emphasizes that these terms cannot be given a simple geometric interpretation, as in eq. (1.5).

General odd $d \geq 3$:

As already noted in footnote 6, the logarithmic divergence in eq. (2.17) vanishes for odd d since $\zeta_{\mathbb{S}^d}^{(0)}(0) = 0$. Hence, evaluating the finite term in the Rényi entropy using eqs. (2.4), (2.17), (2.18) and (2.25) yields:

$$S_{\alpha,finite}^{(0)} = (-1)^{\frac{d+2}{2}} \frac{1+\alpha}{24\alpha} \text{Vol}(\mathbb{S}^d) \left(\frac{1}{\text{Vol}(\mathbb{H}^d)} \frac{d\zeta_{\mathbb{H}^d}^{(0)}}{dz} \Big|_{z=0} - i \frac{2^{d-2}\Gamma(d/2)}{\pi^{\frac{d-2}{2}} R^d} \int_0^\infty \frac{f(ib-y)dy}{1-e^{2\pi(y-ib)}} \right). \quad (2.33)$$

Substituting eq. (B.4) for $\zeta_{\mathbb{H}^d}^{(0)}$ above, then produces

$$S_{\alpha,finite}^{(0)} = (-1)^{\frac{d}{2}} \frac{1+\alpha}{24\alpha} \frac{\text{Vol}(\mathbb{S}^d)}{\pi^{\frac{d-2}{2}} R^d} \left(\frac{1}{2^{d-1}\Gamma(d/2)} \sum_{k=0}^{(d-1)/2} g_{k,d}^{(0)} b^{2k+1} \frac{\sec(k\pi)}{2k+1} \right. \\ \left. + i 2^{d-2}\Gamma(d/2) \int_0^\infty \frac{f(ib-y)dy}{1-e^{2\pi(y-ib)}} \right), \quad (2.34)$$

⁹Here, we mean odd powers of $1/R$ multiplying \mathcal{A}_Σ , which itself contains a factor of R^3 .

where $g_{0,3}^{(0)} = 0$, $g_{1,3}^{(0)} = 1$ and $g_{k,d}^{(0)}$ for odd $d \geq 5$ are defined by

$$\left[x^2 + \left(\frac{d-3}{2} \right)^2 \right] \prod_{j=0}^{(d-5)/2} (x^2 + j^2) = \sum_{k=0}^{(d-1)/2} g_{k,d}^{(0)} x^{2k} . \quad (2.35)$$

From this expression, it is useful to note that

$$g_{\frac{d-1}{2},d}^{(0)} = 1 \quad \text{and} \quad g_{\frac{d-3}{2},d}^{(0)} = \frac{1}{24}(d-1)(d-2)(d-3) , \quad (2.36)$$

as well as $g_{0,d}^{(0)} = 0$. We may then use these expressions to expand $S_{\alpha,finite}^{(0)}$ in the limit $mR \gg 1$ and in doing so, we find

$$S_{\alpha,finite}^{(0)} = \frac{1+\alpha}{24\alpha} \frac{(-1)^{\frac{D-1}{2}} \pi}{(4\pi)^{\frac{D-2}{2}} \Gamma(D/2)} \mathcal{A}_{\Sigma} \left(m^{D-2} + \frac{(D-2)^2(D-3)(6\xi-1)}{12} \frac{m^{D-4}}{R^2} + \dots \right) \quad (2.37)$$

where as before $\mathcal{A}_{\Sigma} = \text{Vol}(\mathbb{S}^d)$ is the area of the entangling surface. Of course, setting $D = 5$, eq. (2.37) simply produces the first two terms in eq. (2.32). For general D , we may set $\alpha = 1$ to recover the entanglement entropy and we see that the leading term above is the precisely the area term expected in eqs. (1.3) and (1.4). The next to leading term in eq. (2.37) introduces a new universal contribution to the entanglement entropy which matches the form shown in eq. (1.5) with $n = 1$. This contribution can be interpreted as

$$S_{\text{univ}} = \frac{D-2}{2} \left(\xi - \frac{1}{6} \right) \gamma_{D,scalar} \int_{\Sigma} d^{D-2} \sigma \sqrt{h} \mathcal{R}(h) m^{D-4} \quad (2.38)$$

where $\mathcal{R}(h)$ is the Ricci scalar of the metric induced on the entangling surface and the coefficient $\gamma_{D,scalar}$ is precisely that given in eq. (1.4). Note that we should only consider this term for odd $D \geq 5$. Of course, there are several other independent curvature terms which could in general contribute at this order but with an appropriate choice of basis, the extra terms all vanish for the waveguide geometry studied here — see section 4 for further discussion.

Even dimensions

Following our discussion of odd d , we first consider the special value $d = 2$ here and evaluate all UV divergences using eqs. (2.4) and (2.8). Then for general even $d \geq 4$, we apply the approach of ζ -function regularization. This method eliminates power law divergences, while keeping the universal terms, *i.e.*, logarithmic divergences, as well as finite contributions to the Rényi entropy.

$d = 2$ ($D = 4$):

In the limit of coincident points, we get from eq. (2.24)

$$K_{\mathbb{S}^2}^{(0)}(t) = \frac{1}{4\pi R^2} \sum_{l=0}^{\infty} (2l+1) e^{-\frac{l(l+1)}{R^2}t}. \quad (2.39)$$

Applying the Euler-Maclaurin formula, *i.e.*,

$$\sum_{l=0}^{\infty} F(l) \simeq \int_0^{\infty} F(x) + \frac{F(0) + F(\infty)}{2} + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} (F^{(2l-1)}(\infty) - F^{(2l-1)}(0)) , \quad (2.40)$$

leads to the following expansion

$$K_{\mathbb{S}^2}^{(0)}(t) = \frac{1}{4\pi t} + \frac{1}{12\pi R^2} + \mathcal{O}(t). \quad (2.41)$$

Combining this expression with eq. (2.18) yields $K_{\mathcal{M}_\alpha}^{(0)}$ as in eq. (2.11). Then substituting the result into eqs. (2.4) and (2.8) yields

$$\begin{aligned} S_\alpha^{(0)} &= \frac{1+\alpha}{24\alpha} \text{Vol}(\mathbb{S}^2) \int_{\delta^2}^{\infty} \frac{dt}{t} e^{-t(m^2+2\xi/R^2)} K_{\mathbb{S}^2}^{(0)}(t) \\ &= \frac{1+\alpha}{48\pi\alpha} \mathcal{A}_\Sigma \left[\frac{1}{2\delta^2} + \left(m^2 + \frac{6\xi-1}{3R^2} \right) \log(m\delta) + \dots \right] \end{aligned} \quad (2.42)$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{S}^2) = 4\pi R^2$ and ellipsis denotes finite terms. Upon setting $\alpha = 1$, we recover the entanglement entropy and the first term is recognized as the standard ‘area law’ contribution. The logarithmic contribution proportional to m^2 matches the area term given in eq. (1.3) with $D = 4$. With $\xi = 1/6$ and $m^2 = 0$, the theory (2.7) under consideration becomes a conformal scalar. In this case, one can verify that the logarithmic contribution above (which vanishes) matches the expected result from eq. (1.2) for a conformal scalar — see section 4 for further discussion.

General even $d \geq 4$:

According to eqs. (2.4) and (2.17) for even d , the universal term is proportional to $\zeta_{\mathbb{S}^d}^{(0)}(0)$. We can evaluate the latter using eq. (2.25). Further, we see that since $\sin(\pi z)$ vanishes at the origin, the latter expression simplifies to

$$\zeta_{\mathbb{S}^d}^{(0)}(0) = (-1)^{d/2} \frac{\text{Vol}(\mathbb{S}^d)}{\text{Vol}(\mathbb{H}^d)} \zeta_{\mathbb{H}^d}^{(0)}(0). \quad (2.43)$$

Hence, using eq. (B.6) for the scalar ζ -function on \mathbb{H}^d , we find that universal contribution is given by

$$S_{\alpha,univ}^{(0)} = \frac{1+\alpha}{12\alpha} \frac{(-1)^{\frac{d+2}{2}} \text{Vol}(\mathbb{S}^d)}{(4\pi)^{d/2} \Gamma(d/2) R^d} \sum_{k=0}^{(d-2)/2} h_{k,d}^{(0)} \left[\frac{(-b^2)^{k+1}}{k+1} - 4 \int_0^\infty \frac{x^{2k+1}}{e^{2\pi x} + 1} dx \right] \log(m\delta), \quad (2.44)$$

where $h_{0,4}^{(0)} = 1/4$, $h_{1,4}^{(0)} = 1$ and $h_{k,d}^{(0)}$ for even $d \geq 6$ are defined by

$$\left[x^2 + \left(\frac{d-3}{2} \right)^2 \right] \prod_{j=1/2}^{(d-5)/2} (x^2 + j^2) = \sum_{k=0}^{(d-2)/2} h_{k,d}^{(0)} x^{2k}. \quad (2.45)$$

Given this definition, it is useful to note that

$$h_{\frac{d-1}{2},d}^{(0)} = 1 \quad \text{and} \quad h_{\frac{d-3}{2},d}^{(0)} = \frac{1}{24} (d-1)(d-2)(d-3). \quad (2.46)$$

Using the latter two expressions in an expansion of $S_{\alpha,finite}^{(0)}$ in the limit $mR \gg 1$ yields

$$S_{\alpha,univ}^{(0)} = \frac{1+\alpha}{12\alpha} \frac{(-1)^{\frac{D}{2}} \mathcal{A}_\Sigma}{(4\pi)^{\frac{D-2}{2}} \Gamma(D/2)} \left(m^{D-2} + \frac{(D-2)^2(D-3)(6\xi-1)}{12} \frac{m^{D-4}}{R^2} + \dots \right) \log(m\delta). \quad (2.47)$$

As usual $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{S}^d)$ is the area of the entangling surface. Note that setting $D = 4$ (*i.e.*, $d=2$), eq. (2.47) reproduces the universal term calculated above in eq. (2.42). Again with $\alpha = 1$, the above reduces to the entanglement entropy and we see that the leading term is the precisely the area term expected in eqs. (1.3) and (1.4). Similar to the discussion for odd d , The next to leading term in eq. (2.47) introduces a new universal contribution to the entanglement entropy which again matches the form shown in eq. (1.5) with $n = 1$. We can write this contribution as

$$S_{univ} = \frac{D-2}{2} \left(\xi - \frac{1}{6} \right) \gamma_{D,scalar} \int_\Sigma d^{D-2} \sigma \sqrt{h} \mathcal{R}(h) m^{D-4} \log(m\delta) \quad (2.48)$$

where $\mathcal{R}(h)$ is the Ricci scalar of the metric induced on the entangling surface and the coefficient $\gamma_{D,scalar}$ is given in eq. (1.4). Here we should only consider this term for even $D \geq 4$. Of course, this expression is reminiscent of eq. (2.38) for the case of odd d .

If we evaluate the entire expression (2.44) for $d = 4$ ($D = 6$), we obtain

$$S_{\alpha,univ}^{(0)}(d=4) = -\frac{1+\alpha}{\alpha} \frac{\text{Vol}(\mathbb{S}^4)}{192 \pi^2} \left[\frac{1}{2} m^4 + 2(6\xi-1) \frac{m^2}{R^2} + \left(72\xi^2 - 24\xi + \frac{29}{15} \right) \frac{1}{R^4} \right] \log(m\delta). \quad (2.49)$$

This example illustrates that the curvature contributions in eq. (1.5) extend up to $n = d/2$ for even d , as can also be seen by directly examining eq. (2.44). In contrast to the case of odd d , these two equations also show that there are no exponentially suppressed terms in the universal contribution for even d .

2.2 Rényi entropy for a massive fermion

In this section, we construct the partition function for a spin- $\frac{1}{2}$ field living on the Euclidean manifold $\mathcal{M}_\alpha = C_\alpha \times \mathbb{S}^d$ and use this result to evaluate the corresponding Rényi entropy. Our spinor notation is reviewed in Appendix A, whereas the action under consideration is given by

$$S(\psi, \bar{\psi}) = \int_{\mathcal{M}_\alpha} (\bar{\psi} \nabla \psi + m \bar{\psi} \psi) . \quad (2.50)$$

In this case, the massless wave operator appearing in the heat kernel (2.8) is

$$\nabla \cdot \nabla^\dagger = -\nabla^2 , \quad (2.51)$$

which we refer to as the ‘iterated’ Dirac operator, following [32]. Since the Dirac operator is a nondiagonal matrix, the separation of variables in eq. (2.11) is not obvious and so we prove that this equation still holds here in Appendix A. The argument there rests on the structure of the heat kernel for the iterated Dirac operator on \mathbb{S}^d , which we review next.

Let us first consider the case of odd d and further for simplicity, let us assume that one of the points coincides with the north pole of \mathbb{S}^d . In this case, the heat kernel reduces to [32]

$$K_{\mathbb{S}^{2j+1}}^{(1/2)}(t, y) = \hat{U}(y) \cos \frac{\theta}{2} \left(\frac{1}{2\pi R^2} \frac{\partial}{\partial \cos \theta} \right)^j \left(\cos \frac{\theta}{2} \right)^{-1} \sum_{n=-\infty}^{+\infty} (-1)^n \frac{e^{-\frac{\theta_n^2 R^2}{4t}}}{(4\pi t)^{1/2}} , \quad (2.52)$$

where y is an arbitrary point on the sphere. The angle of latitude for this point is designated as θ and then $\theta_n = \theta + 2\pi n$. Finally $\hat{U}(y)$ is the spinor matrix which parallel propagates a spinor from the given point y to the north pole. Similarly, in the case of even d , the heat kernel becomes [32]

$$K_{\mathbb{S}^{2j+2}}^{(1/2)}(t, y) = \hat{U}(y) \cos \frac{\theta}{2} \left(\frac{1}{2\pi R^2} \frac{\partial}{\partial \cos \theta} \right)^j \left(\cos \frac{\theta}{2} \right)^{-1} f_{\mathbb{S}^2}^{(1/2)}(\theta, t), \quad (2.53)$$

where

$$f_{\mathbb{S}^2}^{(1/2)}(\theta, t) = \frac{\sqrt{2} R}{(4\pi t)^{3/2} \cos(\theta/2)} \sum_{n=-\infty}^{+\infty} \int_{\theta}^{\pi} \frac{\phi_n \cos \frac{\phi}{2}}{\sqrt{\cos \theta - \cos \phi}} e^{-\frac{R^2 \phi_n^2}{4t}} d\phi , \quad (2.54)$$

with $\phi_n = \phi + 2\pi n$.

The structure of the spinor matrix $\hat{U}(y)$ can be found in [32]¹⁰ but these details will be not important here because we are only interested in the limit of coincident points.

¹⁰We also refer the interested reader to eq. (A.12).

In this limit, $\hat{U}(y)$ simply reduces to an identity matrix. We might note that if we were considering the heat kernel on \mathbb{S}^d alone, the dimension of this identity matrix would be $2^{\lfloor d/2 \rfloor}$, *i.e.*, the dimension of Dirac spinors in d dimensions. Of course, here \mathbb{S}^d is part of the larger manifold \mathcal{M}_α and so the dimension of $\hat{U}(y)$ is actually $2^{\lfloor D/2 \rfloor}$. However, following the conventions introduced in Appendix A, we treat the spinor trace on the right-hand side of eq. (2.12) as though we separately tracing over the spinor spaces of the two component manifolds. That is, we calculate the two spinor heat kernels on C_α and \mathbb{S}^d separately and then simply take their product in eq. (2.12).

The spinor heat kernel on the cone C_α is readily evaluated as [29, 30]

$$\text{Tr} K_{C_\alpha}^{(1/2)}(t) = -\frac{1}{12\alpha} (1 - \alpha^2) + \alpha \text{Tr} K_{\mathbb{R}^2}^{(1/2)}(t). \quad (2.55)$$

As noted above, this result accounts for the trace over the two-dimensional spinor indices on the cone C_α .

Unfortunately, in the present case, we are unable to apply the ζ -function approach, which would have allowed a systematic evaluation of the universal contributions to the Rényi entropy for general d . In particular, while the spinor ζ -function is known for the hyperbolic space \mathbb{H}^d [32], the spin- $\frac{1}{2}$ counterpart of eq. (2.25) relating these to the desired ζ -functions on \mathbb{S}^d is unavailable. Hence, in the following, we limit ourselves to considering a few special cases, *i.e.*, $d = 1, 2$ and 3 . In each case, the Rényi entropy is determined by simply substituting eq. (2.52) or (2.53), along with eq. (2.55) into eqs. (2.4) and (2.8). Given these expressions above, the generalization of the following results to higher dimensions would be straightforward.

Let us add that we consider fermions on the hyperbolic waveguide $C_\alpha \times \mathbb{H}^d$ in Appendix B. In this case, we can follow the ζ -function approach using the results of [32]. This allowed us to produce results for the Rényi entropy of spin- $\frac{1}{2}$ fields for general d and in particular, gave the general coefficients appearing in eqs. (1.6) and (1.9).

$d = 1$ ($D = 3$):

In this case using eq. (2.52), we find

$$\begin{aligned} S_\alpha^{(1/2)} &= \frac{1 + \alpha}{24\alpha} \pi^{1/2} R \sum_{n=-\infty}^{\infty} (-1)^n \int_{\delta^2}^{\infty} \frac{dt}{t^{3/2}} e^{-tm^2 - \frac{(\pi R n)^2}{t}} \\ &= \frac{1 + \alpha}{12\alpha} \left(\frac{\text{Vol}(\mathbb{S}^1)}{\sqrt{4\pi\delta}} - \log[2 \cosh(\pi m R)] \right). \end{aligned} \quad (2.56)$$

In the large mass limit (*i.e.*, $mR \gg 1$), the finite contribution becomes

$$S_{\alpha, \text{finite}}^{(1/2)} = -\frac{1 + \alpha}{24\alpha} (2\pi R m + 2e^{-2\pi m R} - e^{-4\pi m R} + \dots). \quad (2.57)$$

Hence the leading term has precisely the form expected in eq. (1.3) with $\mathcal{A}_\Sigma = 2\pi R$ and $D = 3$. The entanglement entropy is recovered by substituting $\alpha = 1$, in which case the pre-factor matches that given in eq. (1.6) for $D = 3$. Note that this area term is accompanied by higher order exponential terms, similar to those found in eq. (2.28).

$d = 2$ ($D = 4$):

In the limit of coincident points, eq. (2.53) yields

$$K_{\mathbb{S}^2}^{(1/2)}(t) = f_{\mathbb{S}^2}^{(1/2)}(0, t) = \frac{R}{2(\pi t)^{3/2}} \sum_{n=-\infty}^{+\infty} \int_0^{\frac{\pi}{2}} (\phi + \pi n) \cot \phi e^{-\frac{R^2(\phi + \pi n)^2}{t}} d\phi. \quad (2.58)$$

We may note that the integrand is regular at the lower bound. While $\cot \phi$ has a simple pole at $\phi = 0$, the coefficient of this pole is proportional to $\sum_{n=-\infty}^{+\infty} n e^{-\frac{(R\pi n)^2}{t}}$ which vanishes because the summand is odd in n . Redefining the integration variable $\phi \rightarrow \phi/\sqrt{t}$, we have

$$K_{\mathbb{S}^2}^{(1/2)}(t) = \frac{R}{2\pi^{3/2}t^{1/2}} \sum_{n=-\infty}^{+\infty} \int_0^{\frac{\pi}{2\sqrt{t}}} \left(\phi + \frac{\pi n}{\sqrt{t}}\right) \cot(\sqrt{t}\phi) e^{-R^2\left(\phi + \frac{\pi n}{\sqrt{t}}\right)^2} d\phi. \quad (2.59)$$

From this expression, it is obvious that the UV divergences in eq. (2.8) come entirely from the $n = 0$ term. Any terms with $n \neq 0$ contain an exponential factor $e^{-\pi^2 R^2 n^2/t}$ which smoothes out any potential singularities at $t = \delta^2$. Therefore, to extract the structure of the UV divergences, and in particular, the $\log(m\delta)$ contribution, it is enough to examine only the $n = 0$ term, *i.e.*,

$$K_{\mathbb{S}^2}^{(1/2)}(t) = \frac{R}{2\pi^{3/2}t^{1/2}} \int_0^{\frac{\pi}{2\sqrt{t}}} \phi \cot(\sqrt{t}\phi) e^{-R^2\phi^2} d\phi. \quad (2.60)$$

The remaining terms only contribute to the finite part of the Rényi entropy.

Now we expand the integrand in eq. (2.60) in the vicinity of $t = 0$ and find that it is sufficient to keep only first two terms since the rest do not lead to singularities in the limit $\delta \rightarrow 0$

$$K_{\mathbb{S}^2}^{(1/2)}(t) = \frac{1}{2\pi^{3/2}t} \int_0^\infty \left(1 - \frac{t}{3R^2} x^2 + \dots\right) e^{-x^2} dx + \dots = \frac{1}{4\pi t} - \frac{1}{24\pi R^2} + \dots \quad (2.61)$$

Combining this result with eq. (2.55) to form $K_{\mathcal{M}_\alpha}^{(1/2)}$ and substituting the result into eqs. (2.4) and (2.8), we find

$$\begin{aligned} S_\alpha^{(1/2)} &= \frac{1+\alpha}{12\alpha} \text{Vol}(\mathbb{S}^2) \int_{\delta^2}^\infty \frac{dt}{t} e^{-tm^2} K_{\mathbb{S}^2}^{(1/2)}(t) \\ &= \frac{1+\alpha}{48\pi\alpha} \mathcal{A}_\Sigma \left(\frac{1}{\delta^2} + \left(2m^2 + \frac{1}{3R^2}\right) \log(m\delta) + \dots \right) \end{aligned} \quad (2.62)$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{S}^2) = 4\pi R^2$ and ellipsis denotes finite terms. For $\alpha = 1$ the first term represents the standard ‘area law’ in the entanglement entropy, whereas the next term proportional to m^2 is precisely the $D = 4$ case of eqs. (1.3) and (1.6). Similarly, the term proportional to $1/R^2$ matches eqs. (1.7) and (1.9) with $D = 4$.

$d = 3$ ($D = 5$):

Taking the limit $\theta \rightarrow 0$ with $d = 3$ in eq. (2.52), yields

$$K_{\mathbb{S}^3}^{(1/2)}(t, 0) = \frac{1}{(4\pi t)^{3/2}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{(\pi R n)^2}{t}} \left(1 - \frac{t}{2R^2}\right). \quad (2.63)$$

Combining the above with eq. (2.55) to form $K_{\mathcal{M}_\alpha}^{(1/2)}$ and substituting the result into eqs. (2.4) and (2.8), yields

$$\begin{aligned} S_\alpha^{(1/2)} &= \frac{1+\alpha}{12\alpha} \text{Vol}(\mathbb{S}^3) \int_{\delta^2}^{\infty} \frac{dt}{t} e^{-tm^2} K_{\mathbb{S}^3}^{(1/2)}(t, 0) \\ &= \frac{1+\alpha}{6\alpha} \frac{\text{Vol}(\mathbb{S}^3)}{(4\pi)^{3/2}} \left[\frac{1}{3\delta^3} - \left(m^2 + \frac{1}{2R^2}\right) \frac{1}{\delta} \right] + \frac{1+\alpha}{48\pi^2\alpha} g^{(1/2)}(2\pi R m), \end{aligned} \quad (2.64)$$

where

$$g^{(1/2)}(x) = \frac{x^3}{6} + \frac{\pi^2}{2} x + \pi^2 \log(1 + e^{-x}) + \text{Li}_3(-e^{-x}) + x \text{Li}_2(-e^{-x}). \quad (2.65)$$

Expanding the finite contribution above in the limit $mR \gg 1$, we find

$$S_{\alpha, \text{finite}}^{(0)} = \frac{1+\alpha}{72\pi\alpha} \mathcal{A}_\Sigma \left(m^3 + \frac{3}{4} \frac{m}{R^2} - \frac{3}{2\pi^2} \frac{m}{R^2} e^{-2\pi m R} + \dots \right) \quad (2.66)$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{S}^3)$. Setting $\alpha = 1$ to recover the entanglement entropy, the pre-factor becomes $1/(36\pi)$. In this case, the leading term in this expansion is the area term (1.3) with precisely the coefficient given in eq. (1.6) for $D = 5$. Also, the next term matches eqs. (1.7) with the coefficient given by eq. (1.9) for $D = 5$. Further, the above expansion also reveals contributions with exponential factors $\exp(-2\pi n m R)$, similar to those found previously for odd dimensions.

3 A calculation at strong coupling

In this section, we are going to use gauge/gravity duality to study the universal ‘area’ contribution to the entanglement entropy of $\mathcal{N} = 2^*$ gauge theory [20, 21] at strong

coupling. The latter is a massive deformation of the four-dimensional $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory, which is commonly studied in the AdS/CFT correspondence. So let us begin by giving the field theoretic description of the mass terms which appear in this context. The $\mathcal{N} = 4$ SYM theory includes a gauge field A_μ , four Majorana fermions ψ_a and three complex scalars ϕ_i , all of which are in the adjoint representation of the $U(N)$ gauge group. Now there are two independent ‘mass’ terms which can be used to deform the SYM theory [24]

$$\delta\mathcal{L} = -2 \int d^4x \left[m_b^2 \mathcal{O}_2 + m_f \mathcal{O}_3 \right] \quad (3.1)$$

where

$$\mathcal{O}_2 = \frac{1}{3} \text{Tr} \left(|\phi_1|^2 + |\phi_2|^2 - 2|\phi_3|^2 \right) , \quad (3.2)$$

$$\begin{aligned} \mathcal{O}_3 = & -\text{Tr} \left(i \psi_1 \psi_2 - \sqrt{2} g_{\text{YM}} \phi_3 [\phi_1, \phi_1^\dagger] + \sqrt{2} g_{\text{YM}} \phi_3 [\phi_2^\dagger, \phi_2] + \text{h.c.} \right) \\ & + \frac{2}{3} m_f \text{Tr} \left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \right) . \end{aligned} \quad (3.3)$$

The $\mathcal{N} = 2^*$ gauge theory results when we set $m_b = m_f$ and the effect is to give mass to the $\mathcal{N} = 2$ hypermultiplet comprised of $\phi_{1,2}$ and $\psi_{1,2}$. At this point, we note that the dimension-two operator \mathcal{O}_2 contains an unstable mass term for the scalar ϕ_3 — a typical characteristic of such superconformal primary operators. This negative mass-squared contribution from \mathcal{O}_2 is precisely canceled by the positive contribution in \mathcal{O}_3 when $m_b = m_f$ and hence ϕ_3 is left massless in the supersymmetric theory. In the following, we will not restrict our attention to the supersymmetric theory and instead we set m_b and m_f to independent values. In this case, one may then worry that the resulting theory is unstable, *i.e.*, when $m_b > m_f$. Determining the end-point of this instability is the question of understanding the infrared behaviour of the RG flow induced by the mass terms.¹¹ However, recall that the universal contribution to the entanglement entropy appears with a logarithmic dependence on the UV cut-off, $\log \delta$, as shown in eq. (1.3), since we are studying a four-dimensional gauge theory here. Furthermore, as demonstrated in [2] and as we will explicitly see below, the coefficient of this logarithmic term is only determined by the UV properties of the RG flow and is completely insensitive to the IR details. Hence, any such instability is of no consequence to the following calculations.

¹¹The IR theory can be stabilized by introducing a finite temperature, *e.g.*, [22, 23]. However, such a modification of the IR state would again not modify the coefficient of the logarithmic contribution in the entanglement entropy [2].

The dual holographic theory consists of five-dimensional Einstein gravity coupled to a pair of scalars, α and χ , as described by the following action [20, 21]:

$$I_5 = \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \mathcal{L}_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} [R - 3(\partial\alpha)^2 - (\partial\chi)^2 - \mathcal{V}(\alpha, \chi)] , \quad (3.4)$$

where the potential takes the form

$$\mathcal{V}(\alpha, \chi) = -\frac{4}{L^2} e^{-2\alpha} - \frac{8}{L^2} e^\alpha \cosh \chi + \frac{1}{L^2} e^{4\alpha} \sinh^2 \chi . \quad (3.5)$$

Given the above expression, it is trivial to show $\mathcal{V}(\alpha = 0, \chi = 0) = -12/L^2$ and so we see that L corresponds to the curvature scale of the AdS_5 vacuum solution. As it will be needed below, we also note that in the present conventions, Newton's constant is given by

$$G_5 \equiv \frac{\pi L^3}{2N^2} , \quad (3.6)$$

where N is the rank of the $U(N)$ gauge group in the boundary theory. As is well-known, the AdS/CFT correspondence relates the asymptotic boundary behaviour of the scalars to the couplings and expectation values of the dual operators in the boundary theory. Here, α is dual to the ‘*bosonic*’ mass coupling m_b^2 and the corresponding operator \mathcal{O}_2 , given in eq. (3.2). Similarly, χ is dual to the ‘*fermionic*’ mass m_f and the operator \mathcal{O}_3 in eq. (3.3).

In this holographic context, we follow the now standard approach to calculating entanglement entropy [8, 16]. That is, given a spatial region V in the boundary theory, the entanglement entropy between this region and its complement is given by the following expression evaluated in the bulk spacetime:

$$S(V) = \frac{1}{4G_5} \text{ext}_{\partial v \sim \partial V} [A(v)] . \quad (3.7)$$

Here v is a (three-dimensional) bulk surface extending out to asymptotic infinity such that its asymptotic boundary ∂v matches the ‘entangling surface’ ∂V in the boundary geometry. The symbol ‘ext’ indicates that one should extremize the area over all such surfaces v .¹²

Following [24] (see also [22, 23]), we adopt the following ansatz for the bulk solution

$$\begin{aligned} ds^2 &= \frac{L^2}{z^2} [e^{2A(z)} (-B(z)^2 dt^2 + d\vec{x}^2) + dz^2] \\ \alpha &= \alpha(z) , \quad \chi = \chi(z) . \end{aligned} \quad (3.8)$$

¹²If eq. (3.7) is calculated in a Minkowski signature background, the extremal area is only a saddle point. However, if one first Wick rotates to Euclidean signature, the extremal surface will yield the minimal area.

This solution reduces to simply AdS_5 with $A = 0 = B = \alpha = \chi$. In general, turning on the bulk scalars induces a holographic RG flow which is encoded in the metric function $A(z)$. The function $B(z)$ provides a potential blackening factor in the time component of the metric, which also allows the above ansatz to describe finite temperature situations.

In order to identify the $\log \delta$ term in the entanglement entropy, we will only need to consider the asymptotic or small z region of the bulk solution, which describes the UV behaviour of the dual gauge theory. The desired asymptotic solution was found in [24] and takes the form

$$\begin{aligned}\alpha(z) &= z^2(\alpha_{0,1} \log(z) + \alpha_{0,0}) + O(z^4 \log^2 z), \\ \chi(z) &= z \chi_{0,0} + z^3 \left(\frac{4}{3} \chi_{0,0}^3 \log z + \chi_{2,0} \right) + O(z^5 \log^2 z), \\ A(z) &= -\frac{\chi_{0,0}^2}{3} z^2 + O(z^4 \log^2 z), \\ B(z) &= 1 + B_{4,0} z^4 + O(z^6).\end{aligned}\tag{3.9}$$

Of particular importance, the coefficients, $\alpha_{0,1}$ and $\chi_{0,0}$, correspond to the field theory masses [24]¹³

$$\alpha_{0,1} = \frac{m_b^2}{6} \quad \text{and} \quad \chi_{0,0} = \frac{m_f}{2}.\tag{3.10}$$

Similarly, the coefficients $\alpha_{0,0}$ and $\chi_{2,0}$ are related to the expectation values of \mathcal{O}_2 and \mathcal{O}_3 , respectively, while $B_{4,0}$ yields the energy density. We might add that the exact solution describing the supersymmetric flow (with $m_f = m_b = m$) is known [20] and the corresponding asymptotic expansion then becomes [22, 23],

$$\begin{aligned}\alpha(z) &= \frac{m^2 z^2}{6} \left(\log\left(\frac{mz}{2}\right) + \frac{1}{2} \right) + \mathcal{O}(m^4 z^4), \\ \chi(z) &= \frac{m}{2} z + \frac{m^3}{6} z^3 \left(\log\left(\frac{mz}{2}\right) + \frac{1}{4} \right) + \mathcal{O}(m^4 z^4), \\ A(z) &= -\frac{1}{12} m^2 z^2 + \mathcal{O}(m^4 z^4), \quad B(z) = 0.\end{aligned}\tag{3.11}$$

Turning now to the entanglement entropy, our goal is to compare our holographic results to those found at weak coupling. To facilitate this comparison, we must perform the holographic calculation for a ‘waveguide’ geometry analogous to those considered in the previous section. Given the above ansatz (3.8), the boundary geometry is just flat space, but if two of the spatial coordinates are periodic, this geometry can be

¹³With respect to conventions in [24], we have already extracted the L dependence in the metric.

interpreted as a toroidal waveguide $\mathbb{R}^2 \times \mathbb{T}^2$. That is, we will identify the spatial coordinates, x^1 and x^2 , with a period H where $H \gg 1/m_b, 1/m_f$. The entangling surface Σ is then chosen as $x^3 = 0$ (and $t = 0$). According to eq. (3.7), we must find the extremal bulk surface that connects to this entangling surface as $z \rightarrow 0$. However, the high degree of symmetry here dictates that the extremal surface will simply fall straight into the bulk geometry. That is, the desired bulk surface is simply given by $v = \{x^3 = 0, t = 0\}$. Next we must evaluate the area of this surface, however, as usual, the latter must be regulated by cutting off the integral at the UV regulator surface $z = \delta$. We are particularly interested in identifying any contributions proportional to $\log \delta$ and so it is sufficient to consider the asymptotic solution (3.9) in calculating the area.

$$\begin{aligned}
A(v) &= \int dx^1 dx^2 dz \sqrt{g_{11} g_{22} g_{zz}} = H^2 \int_{\delta} dz \frac{L^3}{z^3} e^{2A(z)} \\
&\simeq H^2 L^3 \int_{\delta} \frac{dz}{z^3} \left(1 - \frac{2}{3} \chi_{0,0}^2 z^2 + \mathcal{O}(z^4 \log^2 z) \right) \\
&\simeq \frac{H^2 L^3}{2\delta^2} + \frac{2}{3} H^2 L^3 \chi_{0,0}^2 \log \delta + \mathcal{O}(\delta^2 \log^2 \delta).
\end{aligned} \tag{3.12}$$

Now we may combine this result with eqs. (3.6) and (3.10) to write the entanglement entropy as

$$S_{\text{EE}} = \frac{A(v)}{4G_5} \simeq \frac{N^2 \mathcal{A}_{\Sigma}}{4\pi\delta^2} + \frac{N^2 \mathcal{A}_{\Sigma}}{12\pi} m_f^2 \log(m_f \delta) + \dots \tag{3.13}$$

where $\mathcal{A}_{\Sigma} = H^2$ is the area of the entangling surface in the boundary theory. Of course, the leading contribution here is the expected ‘area law’ contribution, which does not yield any universal information. The next term yields the desired logarithmic term, with the same general form (1.3) as found from free field calculations. We may observe that, as is already evident in eq. (3.12), neither of these UV divergent terms depends on the higher order coefficients in the asymptotic expansion (3.9). Hence as expected [2], our calculation explicitly shows that these contributions to S_{EE} are insensitive to the IR details of the holographic RG flow. Of course, we may also observe that only the fermionic mass m_f appears in the logarithmic term and the result does not depend on the bosonic mass m_b .

While the above result applies to the gauge theory at strong coupling, we can calculate the analogous contribution to the entanglement entropy in the weak coupling limit. In this case, we simply consider the free field limit and apply the results given in eqs. (1.4) and (1.6) in eq. (1.3). If we first turn to \mathcal{O}_2 in eq. (3.2), we see that there are three complex scalars which acquire masses. Since each of these fields is in the adjoint representation of the $U(N)$ gauge group, each ϕ_i effectively contributes $2N^2$

real scalars in the free field limit. Hence using eq. (1.4), the total coefficient in the logarithmic contribution becomes

$$\gamma_{4,b} = \frac{1}{24\pi} \sum 2N^2 m_i^2 = \frac{N^2}{12\pi} \left(\frac{m_b^2}{3} + \frac{m_b^2}{3} - \frac{2m_b^2}{3} \right) = 0. \quad (3.14)$$

Therefore in the weak coupling limit, the bosonic mass term does not in fact contribute to this universal area term in the entanglement entropy. Remarkably, this agrees with our strong coupling result (3.13), in that the bosonic mass m_b does not appear in this expression.

Next, we consider the contribution of \mathcal{O}_3 in the free field limit. In this case, all three scalars again acquire masses and the two fermions ψ_1 and ψ_2 also acquire a mass. In the massless limit, ψ_1 and ψ_2 are two independent Weyl fermions but here they combine as a single massive Dirac fermion. Again the latter are in the adjoint representation and so effectively we have N^2 Dirac fermions in the free field limit. Hence combining the results in eqs. (1.4) and (1.6), the total coefficient in the logarithmic contribution becomes

$$\gamma_{4,f} = \frac{1}{12\pi} N^2 m_f^2 + \frac{1}{24\pi} 2N^2 \left(\frac{2m_f^2}{3} + \frac{2m_f^2}{3} + \frac{2m_f^2}{3} \right) = \frac{N^2}{4\pi} m_f^2. \quad (3.15)$$

Comparing the above with the strong coupling result (3.13), we see that this coefficient is larger by a factor of three than that appearing at strong coupling. We can combine eqs. (3.14) and (3.15) to evaluate the coefficient in the supersymmetric theory with $m_f = m_b = m$:

$$\gamma_{4,susy} = \gamma_{4,f}(m_f = m) + \gamma_{4,s}(m_b = m) = \frac{N^2}{4\pi} m^2. \quad (3.16)$$

Similarly, this choice of masses does not effect the holographic result (3.13) and so the discrepancy observed above between the coefficients in the strong and weak coupling limits extends to the supersymmetric case.

Note that the strong coupling coefficient seems to match the contribution of the fermions alone at weak coupling. That is, we would have found agreement between the two limits if the ‘fermionic’ mass term \mathcal{O}_3 only gave mass to the fermions ψ_1 and ψ_2 . However, as is evident from eq. (3.3), this dimension-three operator also contains a mass term for all three scalars. The latter is somewhat unusual as it is a ‘coupling-dependent’ correction induced at finite mass, *i.e.*, $m_f \mathcal{O}_3$ contains a contribution of order m_f^2 . However, the presence of the additional interactions in eq. (3.3) is dictated by the supersymmetry algebra and the global $SO(6)_R$ symmetry of the $\mathcal{N} = 4$ theory [24]. Further, one can directly detect these scalar masses [34] by uplifting the asymptotic

solution (3.9) to ten dimensions [22] and then examining the potential felt by a probe D3-brane, *e.g.*, [21]. Of course, we should also note that beyond the fermion and scalar mass terms, \mathcal{O}_3 also contains trilinear couplings between the hypermultiplet scalars $\phi_{1,2}$ and the gauge multiplet scalar ϕ_3 . Hence m_f does not simply parameterize the masses of various fields but also plays a role as the coupling of a new cubic potential term in the interacting theory. The latter may well be the source of the discrepancy observed above.

4 Discussion

In this paper, we studied the entanglement entropy for a variety of field theories on waveguide geometries with a spherical or a hyperbolic cross-section. Our calculations confirmed the appearance of a universal area term (1.3), which was first identified in [1], as well as reproducing the precise coefficient (1.4) for a free massive scalar field. Our analysis of the scalar fields in section 2.1 also included a nonminimal curvature coupling $\frac{1}{2}\xi R\phi^2$ and we found the previous area term is not affected by this new coupling. In section 2.2, we also considered a free massive fermion and we identified the same universal area terms as in eq. (1.3) with the coefficient given in eq. (1.6). This reproduces a result given previously in [19].

Curvature contributions:

By considering waveguide geometries with a curved cross-section, we were also able to consider new curvature contributions in the entanglement entropy of the schematic form shown in eq. (1.5). Such terms were first identified with holographic techniques [2], however, we can now present a clearer understanding of the origin of such contributions to entanglement entropy, following the perspective given in, *e.g.*, [35–37]. As noted in the introduction, the calculation of entanglement (or Rényi) entropy in QFT generically yields a UV divergent answer because the result is dominated by the short distance correlations in the vicinity of the entangling surface Σ . Hence the calculation must be regulated by introducing a short distance cut-off δ , and the result typically contains a series of power law divergences, as shown in eq. (1.1) for a QFT in D spacetime dimensions.¹⁴ Of course, the leading contribution has a geometric structure, in that it corresponds to the famous ‘area law’ term [4] with:

$$c_2 = \int_{\Sigma} d^{D-2}\sigma \sqrt{\gamma} d_2 = d_2 \mathcal{A}_{\Sigma}, \quad (4.1)$$

¹⁴Of course, with even D , the c_D term may appear with a logarithmic divergence.

where γ_{ab} is the induced metric on the entangling surface. However, if we are working with a covariant regulator (in a relativistic QFT) and assuming the short-distance cut-off is much smaller than any scale defined by the couplings of the QFT, *i.e.*, $\delta\mu_i \ll 1$, then in fact all of the coefficients of divergent terms in this expansion (1.1) exhibit a similar geometric structure. For example, the second coefficient may be written as

$$c_4 = \int_{\Sigma} d^{D-2}\sigma \sqrt{\gamma} \left[d_{4,1} \mathcal{R}(\gamma) + d_{4,2} R^{ij} \tilde{g}_{ij}^{\perp} + d_{4,3} R^{ijkl} \tilde{g}_{ik}^{\perp} \tilde{g}_{jl}^{\perp} \right. \\ \left. + d_{4,4} K_a^{\hat{i}b} K_b^{\hat{i}a} + d_{4,5} K_a^{\hat{i}a} K_b^{\hat{i}b} \right], \quad (4.2)$$

where, *e.g.*, $\mathcal{R}(\gamma)$ denotes to the intrinsic Ricci scalar of the entangling surface.¹⁵ This geometric character of the coefficients naturally follows from the fact that the UV divergences are all local.

The dimensionless coefficients $d_{2k,a}$ above will of course depend on the detailed structure of the underlying QFT. However, because the $d_{2k,a}$ are dimensionless, we can write their dependence on any mass scale μ_i in the QFT in terms of the dimensionless combination $\mu_i\delta$, *i.e.*, $d_{2k,a} = d_{2k,a}(\mu_i\delta)$. Unfortunately, the coefficients appearing in the expansion above are scheme dependent. Clearly, if we shift $\delta \rightarrow \alpha\delta$, we find $d_{2k,a} \rightarrow \hat{d}_{2k,a} = \alpha^{2k-D} d_{2k,a}(\alpha\mu_i\delta)$. Hence the regulator dependence here comes both from the implicit dependence on mass scales in the QFT and the ‘classical’ engineering dimension of the individual coefficients c_{2k} . Of course, the latter reasoning can be evaded in certain special circumstances. One well-known example, which was already noted in the introduction is the case of a CFT in an even number of spacetime dimensions [6–10]. Given the underlying field theory is conformal, there are no intrinsic mass scales and in an even spacetime dimension, the c_D term still accompanies a logarithmic divergence. Hence the corresponding coefficients $d_{D,a}$ evade the above scaling argument and provide universal information about the underlying field theory. In fact, as illustrated in eq. (1.2), these coefficients are proportional to the various central charges of the CFT.

Another example, which we might consider, is where the underlying field theory describes some renormalization group flow beginning at a UV fixed point. The theory that is probed by the UV singularities (1.1) in the entanglement entropy is a CFT perturbed by some relevant operators. Now let us consider the special case where the RG flow in the UV is controlled by a single¹⁶ mass scale μ . By assumption $\mu\delta \ll 1$ and so we can express the coefficients in terms of a Taylor series: $d_{2k,a} = \sum_{n=0} d_{2k,a}^{(n)} (\mu\delta)^n$. At this point, it is straightforward to extend the previous discussion to show when the

¹⁵We refer the interested reader to [11] for a full explanation of the notation.

¹⁶With more than one scale, one could carry out the following analysis with the largest scale μ_0 while allowing the coefficients $d_{2k,a}^{(n)}$ to be functions of the ratios μ_i/μ_0 .

coefficient $d_{2k,a}^{(n)}$ with $n = D - 2k$ does not scale with α . Hence, we can expect that this coefficient provides universal information about the underlying RG flow. Ultimately, we may repackage this discussion to see that we have identified possible universal contributions to the entanglement entropy which take the form

$$S_{\text{EE}} \simeq d_{2k,a}^{(D-2k)} \mu^{D-2k} \int_{\Sigma} d^{D-2}\sigma \sqrt{\gamma} [(\text{“curvature”})^{k-1}] , \quad (4.3)$$

which, of course, matches the form of the expressions given previously in eq. (1.5).

Of course, the previous discussion can not be complete as we know that universal contributions can also appear with logarithmic factors $\log(\mu\delta)$, as illustrated in eqs. (1.3) and (1.7). So in general, we must allow for such $\log(\mu\delta)$ factors in the expansion of $d_{2k,a}$ to account for these situations. At this point, we would like to note that holographic techniques were used to study the effect on entanglement entropy of perturbing a strongly coupled CFT with relevant operators in [2]. One of the interesting results there was to show that logarithmic contributions can appear in either even or odd spacetime dimensions, in situations where the operators had large anomalous dimensions. For example, perturbing by an operator with conformal dimension $\Delta = (D + 2)/2$ would generate such a $\log\delta$ contribution in general. Further the holographic calculations in [2] show that such large anomalous dimensions may also introduce unusual powers in the expansion of the coefficients $d_{2k,a}$.

The appearance of universal contributions as in eq. (1.5) were first uncovered with a holographic approach in [2] and with sufficient effort, these holographic calculations allowed their precise geometric form to be identified in a straightforward way. However, this identification is much more difficult in the context of field theory calculations presented in this paper. In fact, the geometric nature of various universal contributions is easily confirmed as follows. First, we observe that for sufficiently large masses, our results are naturally be presented in terms of an expansion in powers of $1/(mR)^2$, as could be anticipated from previous calculations [19, 38, 39]. Of course, this is precisely in agreement with the geometric expansion discussed above or given in eq. (1.5), where the powers of $1/R^{2n}$ correspond to factors of $[\text{“curvature”}]^n$. This identification is further confirmed by repeating the calculations for hyperbolic waveguides in Appendix B. There we found that the same expansion is produced up to the replacement $1/(mR)^2 \rightarrow -1/(mR)^2$, as compared to the spherical waveguides in section 2. Of course, this sign corresponds precisely to the change in the sign of the curvature between these two families of geometries.

While in general extracting the precise geometric form of these contributions is difficult with the heat kernel approach used here, the simplicity of our background geometries allowed us to identify the first such universal contribution. As shown in

eq. (1.7), it is simply an integral over the entangling surface of the Ricci scalar evaluated on this surface. In general, one expects that five independent curvature terms would contribute at this order, as shown in eq. (4.2). However, we have chosen a particular ‘basis’ for these terms such that the expression identified in eq. (1.7) is the only nonvanishing term for the waveguide geometries studied here. To be precise, the last two possible contributions in eq. (4.2) involve an integral of terms quadratic in the extrinsic curvature of the entangling surface. However in the present construction, the extrinsic curvature is precisely zero and so these contributions vanish here. Similarly, the second and third contributions in eq. (4.2) involve the background curvature in the space transverse to the entangling surface. In the present case, this transverse geometry is simply \mathbb{R}^2 and so these contributions are also zero.

The coefficient of the curvature contribution identified in eq. (1.7) is given in eqs. (1.8) and (1.9) for the free scalar and fermion fields, respectively. In the scalar case (1.8), we see that this coefficient depends on the non-minimal coupling ξ . Focussing on this linear ξ dependence, we note that the coefficient is precisely that which would appear if we replaced the mass m in the corresponding area term (1.3) by the effective mass m_0 appearing in the action (2.7), *i.e.*,

$$m_0^{D-2} = (m^2 + \xi \mathcal{R})^{(D-2)/2} = m^{D-2} + \frac{D-2}{2} m^{D-4} \xi \mathcal{R} + \dots \quad (4.4)$$

Here we should keep in mind that these results, *i.e.*, eqs. (2.38) and (2.48), were derived from expressions where we had assumed $mR \gg 1$. Hence it is reasonable to consider the Taylor series expansion above. It seems that this simple dependence originates with the appearance of m_0^2 in the definition (2.26) of the parameter b^2 which characterizes the mass dependence of our heat kernels. We might also note that $D = 4$ seems distinguished in eq. (1.8) or (2.48) in that this contribution vanishes when ξ takes the value for a conformal scalar.

As an aside, we point out here that there are finite contributions to the entanglement (and Rényi) entropy which do not fall into the class of terms discussed above. For example, as shown in eqs. (2.28) or (2.32), our results for the spherical waveguides revealed certain ‘topological’ contributions which are exponentially suppressed. We referred to these terms as topological because there were no analogous contributions for waveguides with a hyperbolic cross-section. However, we also see in, *e.g.*, eq. (2.32), that there are contributions with inverse powers of the mass. Recently, it has been shown that such terms with negative powers of m also have a universal character, as they can be related to the universal contributions appearing in entanglement entropy in higher dimensions [19, 38]. Note that if we examine the universal $\log \delta$ contribution in even D , none of the individual terms in this coefficient have the topological character

or inverse powers of the mass described here. The same is true for the coefficients c_{2k} of the power law divergences in eq. (1.1).

Rényi entropy:

Our calculations also extend the initial work of [1] by presenting expressions for the Rényi entropy, as well as the entanglement entropy. In general, our results for the Rényi entropy have the same structure as the entanglement entropy. In fact, for any of the geometries or field theories which we studied here, we can write

$$S_\alpha = \frac{1 + \alpha}{2\alpha} S_{\text{EE}}. \quad (4.5)$$

Hence in general, there are a variety of universal contributions in S_α which take the same geometric form as described above or in eq. (1.5). Unfortunately, with the relation in eq. (4.5), the Rényi entropy would not provide any new information that is not already available in the entanglement entropy. However, it seems that this simple ‘factorization’ of the Rényi entropy must be an artifact of the simplicity of both the background geometries and the QFT’s studied here. Typically, the Rényi entropy has a more complex dependence on the index α than appears in eq. (4.5) as can be seen, *e.g.*, in the holographic results of [40] or the results for disjoint intervals in [41]. Both of these examples also demonstrate that the Rényi entropy often contains far more information about the underlying field theory than the entanglement entropy alone.

The form of the Rényi entropy in eq. (4.5) is reminiscent of well-known results derived in two dimensions [25]. We can compare our expressions for the Rényi entropy for $D = 2$ ($d = 0$) to those given in [25] as a check of our calculations. In that reference, the authors consider a two-dimensional CFT perturbed by a relevant operator which introduces a correlation length $1/m$. Combining various expressions appearing there, the Rényi entropy becomes

$$S = -\frac{1 + \alpha}{12\alpha} c \log(m\delta), \quad (4.6)$$

where c is the central charge of the CFT defining the UV fixed point. In our approach, the result for a massive scalar can be derived by substituting eq. (2.18) directly into eq. (2.8) and the answer then takes precisely the form given above with $c = 1$ as is appropriate for a free scalar field (in the conventions of [25]). Similarly substituting eq. (2.55) into eq. (2.8) yields a result for a massive Dirac fermion, which again has the form given in eq. (4.6) with $c = 1$. Here, we might note that eq. (2.8) has an extra minus sign in the case of the fermion relative to the scalar. However, eq. (2.55) also has an extra minus sign in comparison to its scalar counterpart and so we obtained the desired match.

Strong coupling:

Previously, ref. [2] used holographic techniques to study the effect on entanglement entropy of perturbing a strongly coupled CFT with relevant operators. In the gravity description, such an operator is dual to a scalar field and the holographic entanglement entropy (3.7) is modified by the backreaction of the scalar on the geometry through Einstein's equations. As noted above, one of the interesting results was that logarithmic contributions could appear in either even or odd spacetime dimensions, in situations where the operators had large anomalous dimensions. For example, perturbing by an operator with conformal dimension $\Delta = (D + 2)/2$ would generate such a $\log \delta$ contribution in general. Another result, which seemed to present a small puzzle, was that an operator with the dimension of a scalar mass term, *i.e.*, $\Delta = D - 2$, generated a $\log \delta$ term in even dimensions but only for $D \geq 6$. The puzzle was then that at weak coupling, *i.e.*, with free field theories, such a logarithmic term appears for a massive scalar [1], as shown in eq. (1.3).

Here in section 3, we considered a particular holographic framework where the AdS/CFT dictionary is very well understood, namely the four-dimensional $\mathcal{N} = 2^*$ gauge theory. In this case, two mass operators with $\Delta = 2$ and 3, as well as their holographic description in the dual gravity theory, are known exactly. The detailed description of the bosonic mass term (3.2) in the boundary theory seems to resolve the puzzle noted above. In particular, the intuition provided by weak coupling is that the coefficient of the $\log \delta$ term is proportional to $\sum m_i^2$ in a case where several scalars acquire masses. However, this sum is precisely zero for the present mass term given in eq. (3.2). Hence rather than a puzzle, we have precise agreement between the weak and strong coupling results in this specific case.

In examining the effect of the ‘fermionic’ mass operator (3.3) on the entanglement entropy, we found a contribution in eq. (3.13) of the general form $\mathcal{A}_\Sigma m_f^2 \log(m_f \delta)$, as expected an analogous weak coupling calculation. Unfortunately, if we compare the precise coefficient found at strong coupling with that appearing in the free field limit, there is a discrepancy by a factor of three. Note that the latter adds contributions from both the fermions and scalars which acquire masses when this operator is introduced. As shown in eq. (3.3), \mathcal{O}_3 also contains a cubic interaction between the scalar fields, which vanishes in the weak coupling limit, *i.e.*, $g_{\text{YM}} \rightarrow 0$. However, in the present context then, m_f does not only parameterize the masses of various fields but it also plays a role as the coupling of a new cubic potential term in the interacting theory. Of course, it is tempting to argue that the latter is the source of the discrepancy observed between the strong and weak coupling results. An interesting extension of the present results then would be to calculate the effect of the cubic interaction on the entanglement

entropy perturbatively when the gauge coupling is small but finite.

We would like to contrast the above discrepancy with the recent results in [42]. There the author found in perturbative calculations, that the effect of the interactions on the entanglement entropy was to properly renormalize the mass appearing in eq. (1.3) so that it corresponded to the physical mass. Beyond the usual difficulties that one would encounter in extending such a calculation to strong coupling, our holographic calculation points out another difficulty in taking this limit. Namely, our strongly coupled boundary theory has no simple (quasi)particle excitations and hence we could not identify the ‘renormalized mass’ from a pole in a two-point function. Clearly, a better understanding is needed to appreciate the precise sense in which the entanglement entropy contributions identified in [1] are universal or alternatively, to unravel the precise information that these terms carry about the underlying field theory. It would be useful study further holographic examples where the precise definition of the relevant operators is known in both the bulk gravity and boundary field theories. Another example, which would be interesting for this purpose, is the Cvetič-Gibbons-Lu-Pope solution [43], which describes an RG flow from a three-dimensional CFT [44] in the UV to a gapped theory in the IR.

Comparison with CFT’s:

As noted previously, certain universal contributions to the entanglement entropy are well known in the case of CFT’s in an even number of spacetime dimensions [6–10]. While our calculations focussed on the universal contributions appearing with masses, they should also yield the expected CFT results in the appropriate limits. Hence it is interesting to compare our expressions with the expected CFT results as a check of our calculations.

The universal contribution for a four-dimensional CFT [9] is given in eq. (1.2). Setting $D = 4$ in section 2, the corresponding entangling surface is $\Sigma = \mathbb{S}^2$. In our four-dimensional waveguide geometry, the extrinsic curvatures vanish, $\mathcal{R}_\Sigma = \mathcal{R}_{\mathbb{S}^2} = 2/R^2$ and $C^{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} = \mathcal{R}_{\mathbb{S}^2}/3 = 2/(3R^2)$. Hence, eq. (1.2) yields

$$S_{\text{univ}} = 4 \left(a - \frac{c}{3} \right) \log(\delta/R) . \quad (4.7)$$

Moreover, the central charges for a massless conformal scalar ($\xi = 1/6$) and a massless fermion are [45]:

$$a = \begin{cases} \frac{1}{360} & \text{for } s = 0, \\ \frac{11}{360} & \text{for } s = 1/2, \end{cases} \quad \text{and} \quad c = \begin{cases} \frac{1}{120} & \text{for } s = 0, \\ \frac{1}{20} & \text{for } s = 1/2. \end{cases} \quad (4.8)$$

Therefore, we finally find

$$S_{\text{univ}}(s=0) = 0, \quad \text{and} \quad S_{\text{univ}}(s=1/2) = \frac{1}{18} \log(\delta/R). \quad (4.9)$$

Now we may compare these results with those derived in section 2. In particular, if we set $m = 0$ and $\xi = 1/6$ (and $\alpha = 1$) in eq. (2.42), we see that the $\log \delta$ term in the entanglement entropy vanishes for a conformal scalar, in agreement with the above result. Similarly if we set $m = 0$ (and $\alpha = 1$) in eq. (2.62), we recover precisely the above expression for the universal contribution of a massless Dirac fermion.

Our waveguide geometry also lends itself to using the approach of [10] to determine the universal contribution for a CFT in any number of dimensions. The only restriction of this latter approach is that the background geometry must have a rotational symmetry in the transverse space around the entangling surface, which is certainly satisfied in the present case. The result for the six-dimensional waveguide $\mathbb{R}^2 \times \mathbb{S}^4$ is given in [11] as

$$S_1 = \frac{9\pi\mathcal{A}_\Sigma}{50R^4} \left(\frac{25}{3\pi^3} A - 17B_1 + 52B_2 - 592B_3 \right) \log(m\delta) \quad (4.10)$$

where A , B_1 , B_2 , and B_3 are the four central charges which appear in the trace anomaly of a general CFT in $D = 6$. These central charges were found in [46] for a massless conformal scalar ($\xi = 1/5$):

$$A = -\frac{5}{3 \cdot 7!}, \quad B_1 = \frac{28}{3(4\pi)^3 7!}, \quad B_2 = -\frac{5}{3(4\pi)^3 7!}, \quad \text{and} \quad B_3 = -\frac{2}{(4\pi)^3 7!}; \quad (4.11)$$

and for a massless Dirac fermion:

$$A = -\frac{191}{3 \cdot 7!}, \quad B_1 = \frac{896}{3(4\pi)^3 7!}, \quad B_2 = \frac{32}{(4\pi)^3 7!}, \quad \text{and} \quad B_3 = -\frac{40}{(4\pi)^3 7!}. \quad (4.12)$$

Thus the universal contribution (4.10) to the entanglement entropy becomes

$$S_{\text{univ}} = \begin{cases} \frac{\pi}{67500} \log(m\delta) & \text{for a massless conformal scalar,} \\ -\frac{11\pi}{2700} \log(m\delta) & \text{for a massless Dirac fermion.} \end{cases} \quad (4.13)$$

To extract the corresponding expression for the fermion from our results, we consider eq. (B.41) for the universal contribution on a hyperbolic waveguide in $D = 6$. Replacing $\text{Vol}(\mathbb{H}^4)$ with $\text{Vol}(\mathbb{S}^4) = 8\pi^3 R^4/15$ and $1/R^2 \rightarrow -1/R^2$ in the subsequent factors, we have the analogous contribution for a spherical waveguide. Then setting $m = 0$ and $\alpha = 1$, we recover precisely the universal contribution for a massless fermion given above in eq. (4.13). The corresponding expression for the conformal scalar can be obtained directly from eq. (2.49) after setting $m = 0$, $\xi = 1/5$ and $\alpha = 1$. Unfortunately the

result in this case is: $S_{\text{univ}} = -\frac{\pi}{13500} \log(m\delta)$. Hence we have a discrepancy of a factor of -5 between our calculation and the corresponding result in eq. (4.13)! Unfortunately, at this point, we have not been able to track down the source of this discrepancy despite checking both approaches in many ways, *e.g.*, verifying the central charges in eq. (4.11) with the heat kernel techniques used here.

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A Separation of variables in the case of spin- $\frac{1}{2}$ fields

The main goal of this appendix is to prove the separation of variables (2.11) in the case of spin- $\frac{1}{2}$ fields. Here we focus on the waveguide geometry $\mathcal{M}_\alpha = C_\alpha \times \mathbb{S}^d$ discussed throughout the main text. However, in Appendix B, we also consider the hyperbolic waveguide $C_\alpha \times \mathbb{H}^d$ and hence we note that the present discussion is equally well applicable to the latter case since the essential ingredient is the product structure of the waveguide geometry. We begin by reviewing our spinor notation. The spinors are associated with the orthonormal frames e_a^μ on \mathcal{M}_α

$$e_a^\mu e_b^\nu g_{\mu\nu} = \delta_{ab} , \quad (\text{A.1})$$

whereas the Clifford algebra in the orthonormal frame is generated by D matrices γ^a , satisfying the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} . \quad (\text{A.2})$$

The dimension of these matrices is $2^{\lfloor \frac{D}{2} \rfloor}$ and the associated $D(D-1)/2$ generators of $SO(D)$ rotations are given by

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] . \quad (\text{A.3})$$

The latter satisfy the standard $SO(D)$ commutation rules

$$[\sigma^{ab}, \sigma^{cd}] = \delta^{bc}\sigma^{ad} - \delta^{ac}\sigma^{bd} - \delta^{bd}\sigma^{ac} + \delta^{ad}\sigma^{bc} , \quad (\text{A.4})$$

while the commutator of σ^{ab} with γ^c is

$$[\sigma^{ab}, \gamma^c] = \delta^{bc} \gamma^a - \delta^{ac} \gamma^b . \quad (\text{A.5})$$

The covariant derivative of a spinor may be written in terms of e_a^μ as follows:

$$\nabla_a = e_a^\mu \nabla_\mu , \quad \nabla_\mu = \partial_\mu + \frac{1}{2} \sigma^{bc} \omega_{\mu bc} , \quad \omega_{\mu bc} = e_b^\nu (\partial_\mu e_{c\nu} - \Gamma_{\nu\mu}^\alpha e_{c\alpha}) , \quad (\text{A.6})$$

where $\Gamma_{\nu\mu}^\alpha$ are the usual Christoffel symbols. With this definition, the following anti-commutation relations can be shown to hold [47]

$$[\nabla_a, \nabla_b] \psi = -\frac{1}{2} R_{abcd} \sigma^{cd} \psi . \quad (\text{A.7})$$

In particular, since the γ^a matrices are covariantly constant, one can use eqs. (A.2) and (A.7) to verify the following identity for the iterated Dirac operator:

$$\not{\nabla} \cdot \not{\nabla}^\dagger = -\not{\nabla}^2 = -(\gamma^a \nabla_a)^2 = -\delta^{ab} \nabla_a \nabla_b + \frac{R}{4} . \quad (\text{A.8})$$

It is the heat kernel of this operator on \mathcal{M}_α that we use in eq. (2.8) to evaluate the partition function in the case of spin- $\frac{1}{2}$ fields.

Since the waveguide geometry has a product structure, it is natural to choose the first two γ -matrices $a = 1, 2$ to construct the Dirac operator on C_α and then the rest are used to build the restriction of Dirac operator to \mathbb{S}^d . Now, we observe that due to the various relations in eqs. (A.2), (A.4) and (A.5), the restriction of Dirac operator to C_α anticommutes with the restriction of Dirac operator to \mathbb{S}^d . Therefore we can split the iterated Dirac operator into separate wave operators on C_α and \mathbb{S}^d .

It is convenient to make this discussion more explicit by choosing the following representation of the Dirac matrices on \mathcal{M}_α :

$$\begin{aligned} \gamma^1 &= \sigma^1 \otimes \mathbb{I}_d , & \gamma^2 &= \sigma^2 \otimes \mathbb{I}_d , \\ \gamma^a &= \sigma^3 \otimes \hat{\gamma}^{a-2} \quad \text{for } a = 3, \dots, D . \end{aligned} \quad (\text{A.9})$$

where σ^i are the standard Pauli matrices while $\hat{\gamma}^a$ and \mathbb{I}_d is the Dirac matrices and unit matrix of the Clifford algebra on \mathbb{S}^d . Hence the latter matrices have dimension $2^{\lfloor d/2 \rfloor} \times 2^{\lfloor d/2 \rfloor}$. With this explicit representation, the Dirac operator takes the form

$$\not{\nabla}_{\mathcal{M}_\alpha} = \not{\nabla}_{C_\alpha} \otimes \mathbb{I}_d + \sigma^3 \otimes \not{\nabla}_{\mathbb{S}^d} \quad (\text{A.10})$$

where $\not{\nabla}_{C_\alpha}$ and $\not{\nabla}_{\mathbb{S}^d}$ are respectively the Dirac operators that would be constructed on C_α and \mathbb{S}^d alone. Similarly, the iterated Dirac operator appearing in the heat kernel becomes

$$-\not{\nabla}_{\mathcal{M}_\alpha}^2 = -\not{\nabla}_{C_\alpha}^2 \otimes \mathbb{I}_d + \mathbb{I}_2 \otimes \left(-\not{\nabla}_{\mathbb{S}^d}^2 \right) . \quad (\text{A.11})$$

Hence it is clear that the heat kernel of the iterated massless Dirac operator on the waveguide $C_\alpha \times \mathbb{S}^d$ can be written as a product of the individual heat kernels, as in eq. (2.11). Hence the desired separation of variables is proved.

Further we note that the spinors on \mathcal{M}_α have dimension $2^{\lfloor D/2 \rfloor} = 2 \cdot 2^{\lfloor d/2 \rfloor}$. With the above construction, we see that the full trace over these spinor indices is properly accounted for by calculating the two spinor heat kernels on C_α and \mathbb{S}^d separately and then simply taking their product in eq. (2.12). That is, we can treat the spinor trace on the right-hand side of eq. (2.12) as though we separately tracing over the spinor spaces of the two component manifolds.

While the desired result has been established, let us make a few more comments. Recall that the construction of the spinor heat kernels \mathbb{S}^d given in eqs. (2.52) and (2.53) involve a spinor matrix \hat{U} which parallel propagates a spinor between the two points in the heat kernel. According to [32], this ‘propagator’ $\hat{U}(x, y)$ on \mathbb{S}^d can be formally written as follows

$$\hat{U}(x, y) = P \exp \frac{1}{2} \int_x^y \omega_{\mu bc}(t) \hat{\sigma}^{bc} v^\mu(t) dt, \quad (\text{A.12})$$

where $\hat{\sigma}^{ab}$ are the $SO(d)$ generators (A.3) for the d -dimensional Clifford algebra, now constructed with the Dirac matrices $\hat{\gamma}^a$. Above, the integration is along the shortest geodesic connecting $x, y \in \mathbb{S}^d$, P is the path-ordering operator and $v^\mu(t) = dx^\mu/dt$ is the tangent vector to the geodesic $x^\mu(t) \in \mathbb{S}^d$. With the representation in eq. (A.9), this propagator becomes $U(x, y) = \mathbb{I}_2 \otimes \hat{U}(x, y)$. Hence, given eq. (A.11), this again demonstrates that the restriction of the iterated Dirac operator to C_α commutes with the heat kernels on \mathbb{S}^d given in eqs. (2.52) and (2.53). This is, of course, necessary for eq. (2.11) to hold.

Finally, let us comment that demonstrating the separation of variables in eq. (2.11) here crucially depends on the fact that we are studying the iterated Dirac operator (2.51) rather than working with the Dirac operator directly. In contrast, the eigenspinors of the ∇ on $C_\alpha \times S^d$ (or H^d) cannot be obtained easily from the lower dimensional eigenspinors on the component manifolds C_α and S^d . While one might construct the heat kernel on \mathcal{M}_α using eigenspinors, clearly such a construction would obscure the desired separation of variables and our discussion above demonstrates that in fact this approach is not needed.

B Waveguides with hyperbolic cross-section

Throughout the main text, we considered the waveguide geometry with a spherical cross-section. In this appendix, \mathbb{S}^d is replaced by a d -dimensional hyperbolic space \mathbb{H}^d

and we show that Rényi entropy is essentially the same with an obvious sign change $R^2 \rightarrow -R^2$ in various curvature contributions.

In the following, we mimic our previous discussion of the case of a spherical cross-section. In particular, massive scalars and fermions will be considered separately, and within each of these, odd and even dimensions d require separate consideration. Further, both the simple regularization of the heat kernel (2.8) and the ζ -function approach (2.17) will be used to evaluate the Rényi entropy. Recall that while the regularization in eq. (2.8) reproduces the whole structure of the Rényi entropy, the ζ -function method (2.17) eliminates the power law divergences and retains only finite and logarithmically divergent terms. The general remarks and formulae of section 2 do not require any modifications apart from obvious replacements $\mathbb{S}^d \rightarrow \mathbb{H}^d$ and $R^2 \rightarrow -R^2$ and will not be repeated in the following discussion. Instead we focus on the evaluation the heat kernel $K_{\mathbb{H}^d}$ and $\zeta_{\mathbb{H}^d}^{(s)}$.

B.1 Rényi entropy for a massive scalar

There is a vast literature which considers the scalar heat kernel on the hyperbolic space \mathbb{H}^d , *e.g.*, see [48]

$$K_{\mathbb{H}^{2n+1}}^{(0)}(t, x, y) = \frac{1}{(4\pi t)^{1/2}} \left(\frac{-1}{2\pi R^2 \sinh \rho} \frac{\partial}{\partial \rho} \right)^n e^{-\frac{n^2 t}{R^2} - \frac{\rho^2 R^2}{4t}}, \quad (\text{B.1})$$

$$K_{\mathbb{H}^{2n+2}}^{(0)}(t, x, y) = e^{-\frac{(2n+1)^2}{4R^2} t} \left(\frac{-1}{2\pi R^2 \sinh \rho} \frac{\partial}{\partial \rho} \right)^n f_{\mathbb{H}^2}^{(0)}(\rho, t), \quad (\text{B.2})$$

where n is or a positive integer; ρ is the geodesic distance between x and y measured in units of R ; and

$$f_{\mathbb{H}^2}^{(0)}(\rho, t) = \frac{\sqrt{2}R}{(4\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{\tilde{\rho} e^{-\frac{R^2 \tilde{\rho}^2}{4t}}}{\sqrt{\cosh \tilde{\rho} - \cosh \rho}} d\tilde{\rho}. \quad (\text{B.3})$$

The scalar ζ -function was computed in [31] where it was shown that for odd $d \geq 3$

$$\zeta_{\mathbb{H}^d}^{(0)}(z) = \frac{R^{2z-d} b^{1-2z} \text{Vol}(\mathbb{H}^d)}{(4\pi)^{d/2} \Gamma(d/2)} \sum_{k=0}^{(d-1)/2} g_{k,d}^{(0)} b^{2k} B(k+1/2, z-k-1/2), \quad (\text{B.4})$$

where B denotes the usual beta function with $B(x, y) = \Gamma(x) \Gamma(y) / \Gamma(x+y)$ and

$$b^2 = R^2 m^2 - \xi d(d-1) + \frac{(d-1)^2}{4}. \quad (\text{B.5})$$

Note that b^2 here in the case of \mathbb{H}^d is the same from its counterpart (2.26) on \mathbb{S}^d with the replacement $R^2 \rightarrow -R^2$. The coefficients $g_{k,d}^{(0)}$ were introduced previously — see

eqs. (2.35) and (2.36), as well as the surrounding discussion. For even $d \geq 4$

$$\zeta_{\mathbb{H}^d}^{(0)}(z) = \frac{R^{2z-d} \text{Vol}(\mathbb{H}^d)}{(4\pi)^{d/2} \Gamma(d/2)} \sum_{k=0}^{(d-2)/2} h_{k,d}^{(0)} \left[b^{2k+2-2z} B(k+1, z-k-1) - 4 \int_0^\infty \frac{x^{2k+1}}{(e^{2\pi x} + 1)(x^2 + b^2)^z} dx \right], \quad (\text{B.6})$$

where the coefficients $h_{k,d}^{(0)}$ are given in eqs. (2.45) and (2.46) (as well as the surrounding discussion). In both eqs. (B.4) and (B.6), one should think of $\text{Vol}(\mathbb{H}^d)$ as a formal regulated volume for the cross-section of the waveguide geometry. While we express our results below in terms of this volume, it may be more appropriate to think in terms of the Rényi entropy density along the entangling surface.

Odd dimensions

Let us assume that $d = 2n + 1$ and take the limit of coincident points in eq. (B.1), then $K_{\mathbb{H}^d}^{(0)}(t, x, x)$ takes the following general form [49]

$$K_{\mathbb{H}^{2n+1}}^{(0)}(t, x, x) = \frac{P_{n-1}^{(0)}(t/R^2)}{(4\pi t)^{n+1/2}} e^{-\frac{n^2 t}{R^2}}. \quad (\text{B.7})$$

where from (B.1) it follows that $P_{n-1}^{(0)}(x)$ is polynomial of degree $n - 1$ with rational coefficients. For $n = 0$, $P_{-1}^{(0)}(x) = 1$ while for $n > 0$, $P_{n-1}^{(0)}(x)$ is polynomial of degree $n - 1$ with rational coefficients:

$$P_{n-1}^{(0)}(x) = \sum_{j=0}^{n-1} a_{j,n-1}^{(s)} x^j. \quad (\text{B.8})$$

For example, let us write out the first few polynomials

$$\begin{aligned} P_0^{(0)}(x) &= 1, \\ P_1^{(0)}(x) &= 1 + \frac{2}{3}x, \\ P_2^{(0)}(x) &= 1 + 2x + \frac{16}{15}x^2, \\ P_3^{(0)}(x) &= 1 + 4x + \frac{28}{5}x^2 + \frac{96}{35}x^3. \end{aligned} \quad (\text{B.9})$$

As one may surmise from these examples, $a_{0,n-1}^{(0)} \equiv 1$ for $n \geq 0$. To simplify the following discussion, we also denote $a_{j,-1}^{(0)} = 0$ for $j > 0$. We may evaluate the Rényi entropy by combining the above heat kernels (B.7) with the usual expressions given in the main text (*i.e.*, eqs. (2.4), (2.8), (2.12) and (2.18)), which yields

$$S_\alpha^{(0)} = \frac{1 + \alpha}{24 \alpha} \text{Vol}(\mathbb{H}^{2n+1}) \int_{\delta^2}^\infty \frac{dt}{t} \frac{P_{n-1}^{(0)}(t/R^2)}{(4\pi t)^{n+1/2}} e^{-\frac{b^2 t}{R^2}}. \quad (\text{B.10})$$

We can easily evaluate the first few divergent terms in this expression as

$$S_{\alpha,div}^{(0)} = \frac{1+\alpha}{12\alpha} \frac{\text{Vol}(\mathbb{H}^d)}{(4\pi)^{d/2}} \frac{1}{\delta^d} \left(\frac{1}{d} + \frac{a_{1,\frac{d-3}{2}} - b^2}{(d-2)} \frac{\delta^2}{R^2} + \dots \right) \quad (\text{B.11})$$

Here we have used that in the vicinity of $t = 0$, $P_{n-1}^{(0)}(t/R^2) \simeq 1$. As expected, we see that the leading divergence obeys the expected ‘area law.’ Continuing to higher orders in t , one can reconstruct all the UV divergences. We note that the structure of these divergences matches those for the case of \mathbb{S}^d up to an obvious sign flip in the background curvature $R^2 \rightarrow -R^2$, *e.g.*, one might compare the above expression with $n = 1$ ($d = 3$) to eq. (2.31).

A simple approach to evaluate the finite terms in $S_{\alpha}^{(0)}$ is to proceed in the spirit of dimensional regularization by setting $\delta = 0$ but treating d as unspecified variable in the integration over t in eq. (B.10). This yields

$$S_{\alpha,finite}^{(0)} = \frac{1+\alpha}{24\alpha} \frac{\text{Vol}(\mathbb{H}^d)}{(4\pi R^2)^{d/2}} \sum_{j=0}^{\frac{d-3}{2}} a_{j,\frac{d-3}{2}}^{(0)} \Gamma(j - d/2) b^{d-2j} . \quad (\text{B.12})$$

This expression is also valid for $d = 1$ if we omit the sum and substitute $j = 0$. In the limit $mR \gg 1$, the leading behaviour in eq. (B.12) is given by

$$S_{\alpha,finite}^{(0)} = \frac{1+\alpha}{24\alpha} \frac{(-1)^{\frac{D-1}{2}} \pi}{(4\pi)^{\frac{D-2}{2}} \Gamma(D/2)} \mathcal{A}_{\Sigma} \left(m^{D-2} + \frac{D-2}{2} \frac{m^{D-4}}{R^2} \right. \\ \left. \times \left[\frac{(D-3)^2}{4} - (D-2)(D-3)\xi - a_{1,\frac{D-5}{2}}^{(0)} \right] + \dots \right) \quad (\text{B.13})$$

where $\mathcal{A}_{\Sigma} = \text{Vol}(\mathbb{H}^d)$ and we have again used $a_{0,(D-3)/2}^{(0)} \equiv 1$. Further we have simplified the above result with the following identity

$$\Gamma(1 - D/2) \Gamma(D/2) = \frac{\pi}{\sin(\frac{\pi D}{2})} = (-1)^{\frac{D-1}{2}} \pi \quad (\text{B.14})$$

where the last equality applies because we are only considering odd D . If we compare this result with eqs. (1.3) and (1.4), we see that the expression gives precisely the expected area term. We can also compare this result with eq. (2.37) for the spherical waveguide with odd D . In this case, the second contribution above matches the corresponding term in eq. (2.37) with $R^2 \rightarrow -R^2$ if

$$a_{1,\frac{D-5}{2}}^{(0)} = \frac{(D-3)(D-5)}{12} , \quad (\text{B.15})$$

which matches the linear coefficients in the examples given in eq. (B.9). Of course, the $R^2 \rightarrow -R^2$ behaviour is precisely that expected of a curvature contribution, as given in eq. (2.38).

Alternatively we can apply the ζ -function method (2.17) to derive $S_{\alpha,finite}^{(0)}$. According to eq. (B.4), $\zeta_{\mathbb{H}^d}^{(0)}(0) = 0$ and therefore, from eqs. (2.4) and (2.18), we find

$$S_{\alpha,finite}^{(0)} = \frac{1+\alpha}{24\alpha} \text{Vol}(\mathbb{H}^d) \left. \frac{d\zeta_{\mathbb{H}^d}^{(0)}}{dz} \right|_{z=0} \quad (\text{B.16})$$

Comparing this result to eq. (2.33), we see that the differences between the \mathbb{H}^d and \mathbb{S}^d cases are not accounted for by a change in the sign of the curvature alone. While the first term in eq. (2.33) matches that the (entire) result above with the replacement $R^2 \rightarrow -R^2$, there is also an extra integral contributing on the right hand side of eq. (2.33). Recall that this integral vanishes in the limit $mR \rightarrow \infty$, however, for large but finite mR , it contributes exponentially suppressed terms to $S_{\alpha,finite}^{(0)}$, *e.g.*, see (2.32). In the main text, we speculated that these exponential contributions probe the topology of the waveguide geometry. In this case, it would be natural that they vanish here where the \mathbb{H}^d cross-section is topologically trivial. Lastly, we note that we may evaluate eq. (B.16) using eq. (B.6) and compare the result with (B.13). Agreement in this comparison again yields eq. (B.15).

Even dimensions

Below we evaluate the structure of all of UV divergences for the particular case of $d = 2$. Then using ζ -function approach, we evaluate (only) the universal contributions for any even $d \geq 4$.

$d = 2$ ($D = 4$):

In this case, we substitute $n = 0$ in eq. (B.2) and take the limit of coincident points

$$\begin{aligned} K_{\mathbb{H}^2}^{(0)}(t, x, x) &= e^{-\frac{t}{4R^2}} f_{\mathbb{H}^2}^{(0)}(0, t) = \frac{e^{-\frac{t}{4R^2}}}{2\pi^{\frac{3}{2}} R \sqrt{t}} \int_0^\infty \frac{x e^{-x^2}}{\sinh\left(\frac{\sqrt{t}x}{R}\right)} dx \\ &= \frac{e^{-\frac{t}{4R^2}}}{2\pi^{\frac{3}{2}} t} \int_0^\infty dx e^{-x^2} \left(1 - \frac{t}{6R^2} x^2 + \dots\right) = \frac{e^{-\frac{t}{4R^2}}}{4\pi t} \left(1 - \frac{t}{12R^2} + \dots\right), \end{aligned} \quad (\text{B.17})$$

This result is then used as usual to evaluate the Rényi entropy, yielding

$$S_\alpha^{(0)} = \frac{1+\alpha}{96\pi\alpha} \text{Vol}(\mathbb{H}^2) \int_{\delta^2}^{\infty} \frac{dt}{t^2} e^{-\frac{t}{R^2} (1/4 - 2\xi + R^2 m^2)} \left(1 - \frac{t}{12R^2} + \dots\right) \quad (\text{B.18})$$

$$= \frac{1+\alpha}{48\pi\alpha} \text{Vol}(\mathbb{H}^2) \left[\frac{1}{2\delta^2} + \left(m^2 - \frac{6\xi - 1}{3R^2}\right) \log(m\delta) + \dots \right]. \quad (\text{B.19})$$

Hence the leading divergence reveals the expected ‘area law’ behaviour, while with $\alpha=1$, the second term is precisely the result given in eqs. (1.3) and (1.4) for $D = 4$. Further, this result agrees with the \mathbb{S}^2 counterpart (2.42) with the usual $R^2 \rightarrow -R^2$ replacement.

General even $d \geq 4$.

While we could determine the entire pattern of divergences in the Rényi entropy in higher even dimensions by extending the method considered above, here we focus our attention on only the universal contributions. The latter are readily evaluated using the ζ -function approach (2.17) instead.

Evaluating scalar ζ -function (B.6) at $z = 0$ and substituting the result into eqs. (2.4) and (2.17), we find that universal contribution to the Rényi entropy is given by

$$S_\alpha^{(0)} = \frac{1+\alpha}{12\alpha} \frac{\text{Vol}(\mathbb{H}^d)}{(4\pi)^{d/2} \Gamma(d/2) R^d} \sum_{k=0}^{(d-2)/2} h_{k,d}^{(0)} \left[4 \int_0^\infty \frac{x^{2k+1}}{e^{2\pi x} + 1} dx - \frac{(-b^2)^{k+1}}{k+1} \right] \log(m\delta). \quad (\text{B.20})$$

Expanding in the limit $mR \gg 1$, yields

$$S_{\alpha, \text{univ}}^{(0)} = \frac{1+\alpha}{12\alpha} \frac{(-1)^{\frac{D}{2}} \mathcal{A}_\Sigma}{(4\pi)^{\frac{D-2}{2}} \Gamma(D/2)} \left(m^{D-2} - \frac{(D-2)^2(D-3)(6\xi-1)}{12} \frac{m^{D-4}}{R^2} + \dots \right) \log(m\delta), \quad (\text{B.21})$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{H}^d)$. The above results are in full agreement with their counterparts for the spherical waveguide in eqs. (2.44) and (2.47) provided that we replace $R^2 \rightarrow -R^2$.

B.2 Rényi entropy for a massive fermion

The spinor heat kernel on the hyperbolic space was evaluated in [32] as

$$K_{\mathbb{H}^{2n+1}}^{(1/2)}(t, x, y) = \hat{U}(x, y) \cosh \frac{\rho}{2} \left(\frac{-1}{2\pi R^2} \frac{\partial}{\partial \cosh \rho} \right)^n \left(\cosh \frac{\rho}{2} \right)^{-1} \frac{e^{-\frac{\rho^2 R^2}{4t}}}{(4\pi t)^{1/2}}, \quad (\text{B.22})$$

$$K_{\mathbb{H}^{2n+2}}^{(1/2)}(t, x, y) = \hat{U}(x, y) \cosh \frac{\rho}{2} \left(\frac{-1}{2\pi R^2} \frac{\partial}{\partial \cosh \rho} \right)^n \left(\cosh \frac{\rho}{2} \right)^{-1} f_{\mathbb{H}^2}^{(1/2)}(\rho, t), \quad (\text{B.23})$$

where x, y are two arbitrary points of the hyperbolic space; $n = 0, 1, 2, \dots$; ρ is the geodesic distance between x and y in units of R ; the matrix $\hat{U}(x, y)$ is the parallel spinor propagator from x to y ; and

$$f_{\mathbb{H}^2}^{(1/2)}(\rho, t) = \frac{\sqrt{2} R}{(4\pi t)^{3/2} \cosh(\rho/2)} \int_{\rho}^{\infty} \frac{\tilde{\rho} \cosh \frac{\tilde{\rho}}{2} e^{-\frac{R^2 \tilde{\rho}^2}{4t}}}{\sqrt{\cosh \tilde{\rho} - \cosh \rho}} d\tilde{\rho} . \quad (\text{B.24})$$

Alternatively, one can use ζ -function method (2.17) to evaluate the partition function Z_{α} . The spinor ζ -function on \mathbb{H}^d was computed in [32]. For odd $d \geq 3$

$$\zeta_{\mathbb{H}^d}^{(1/2)}(z) = \frac{2^{[\frac{d}{2}]} R^{1-d} m^{1-2z} \text{Vol}(\mathbb{H}^d)}{(4\pi)^{d/2} \Gamma(d/2)} \sum_{k=0}^{(d-1)/2} g_{k,d}^{(1/2)} (Rm)^{2k} B(k+1/2, z-k-1/2) , \quad (\text{B.25})$$

where $g_{k,d}^{(1/2)}$ are defined by

$$\prod_{j=1/2}^{(d-2)/2} (x^2 + j^2) = \sum_{k=0}^{(d-1)/2} g_{k,d}^{(1/2)} x^{2k} . \quad (\text{B.26})$$

On the other hand, for even $d \geq 2$

$$\begin{aligned} \zeta_{\mathbb{H}^d}^{(1/2)}(z) = \frac{R^{2z-d} \text{Vol}(\mathbb{H}^d)}{(2\pi)^{d/2} \Gamma(d/2)} \sum_{k=0}^{(d-2)/2} h_{k,d}^{(1/2)} \left[(mR)^{2k+2-2z} B(k+1, z-k-1) \right. \\ \left. + 4 \int_0^{\infty} \frac{x^{2k+1}}{(e^{2\pi x} - 1)(x^2 + m^2 R^2)^z} dx \right] , \quad (\text{B.27}) \end{aligned}$$

where $h_{0,2}^{(1/2)} = 1$ and $h_{k,d}^{(1/2)}$ for even $d > 2$ are defined by

$$\prod_{j=1}^{(d-2)/2} (x^2 + j^2) = \sum_{k=0}^{(d-2)/2} h_{k,d}^{(1/2)} x^{2k} . \quad (\text{B.28})$$

Clearly the ζ -function approach (2.17) is the most efficient way to determine the universal contributions in the Rényi entropy (2.4). However, as stressed previously, this method eliminates all of the power law divergences, while the sharp cut-off in eq. (2.8) allows us to keep track of the entire structure of the UV divergences appearing the Rényi entropy. Therefore, in what follows, we exploit both approaches to shed light on the structure of the Rényi entropy in the case of the spin- $\frac{1}{2}$ field.

Odd dimensions

It follows from eq. (B.22) that for $d = 2n + 1$, $K_{\mathbb{H}^d}^{(1/2)}(x, x, t)$ takes the following general form

$$K_{\mathbb{H}^{2n+1}}^{(1/2)}(x, x, t) = \frac{P_n^{(1/2)}(t/R^2)}{(4\pi t)^{n+1/2}} \mathbb{I}_{n+1}, \quad (\text{B.29})$$

where \mathbb{I}_{n+1} is a unit matrix of dimension $n + 1$. This matrix is the remnant of the spinor propagator $\hat{U}(x, y)$ in the limit of coincident points. $P_n^{(1/2)}(x)$ is polynomial of degree n with rational coefficients

$$P_n^{(1/2)}(x) = \sum_{j=0}^n a_{j,n}^{(1/2)} x^j, \text{ for } n \geq 0. \quad (\text{B.30})$$

The first few of these polynomials are given by

$$\begin{aligned} P_0^{(1/2)}(x) &= 1, \\ P_1^{(1/2)}(x) &= 1 + \frac{1}{2}x, \\ P_2^{(1/2)}(x) &= 1 + \frac{5}{3}x + \frac{3}{4}x^2, \\ P_3^{(1/2)}(x) &= 1 + \frac{7}{2}x + \frac{259}{60}x^2 + \frac{15}{8}x^3. \end{aligned} \quad (\text{B.31})$$

By definition $a_{0,n}^{(1/2)} \equiv 1$ and as we confirm below, in general,

$$a_{1, \frac{D-3}{2}}^{(1/2)} = \frac{(D-2)(D-3)}{12}. \quad (\text{B.32})$$

Eq. (B.29) leads to the following expression for the Rényi entropy,

$$S_\alpha^{(1/2)} = \frac{2^n(1+\alpha)}{24\alpha} \text{Vol}(\mathbb{H}^{2n+1}) \int_{\delta^2}^\infty \frac{dt}{t} \frac{P_n^{(1/2)}(t/R^2)}{(4\pi t)^{n+1/2}} e^{-m^2 t}. \quad (\text{B.33})$$

The integral above is divergent in the vicinity of $t = 0$. However, since $P_n^{(1/2)}(x)$ is a polynomial of degree n and the denominator contains a half-integer power of t , all the divergences are simply power-law (and thus non-universal). Using $a_{0,n}^{(1/2)} = 1$, the leading contributions take the form

$$S_{\alpha, div}^{(1/2)} = \frac{(1+\alpha)}{12\alpha} \frac{\sqrt{\pi} \mathcal{A}_\Sigma}{(D-2)(2\pi)^{\frac{D-1}{2}}} \left(\frac{1}{\delta^{D-2}} - \frac{D-2}{D-4} \left(m^2 - \frac{a_{1, \frac{D-3}{2}}^{(1/2)}}{R^2} \right) \frac{1}{\delta^{D-4}} + \dots \right), \quad (\text{B.34})$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{H}^{D-2})$. Hence, the leading divergence exhibits the expected ‘area law’ behaviour. Continuing the process of expanding the integrand in eq. (B.33) near $t = 0$,

all power law divergences can be evaluated for any fixed value of D . We can compare the above expression to the results for a spherical waveguide for $D = 3$ and 5 given eqs. (2.56) and (2.64), respectively. Then up to expected substitution $R^2 \rightarrow -R^2$ in the curvature contributions, we see that the divergences in these two different geometries agree.

To compute the finite part of the Rényi entropy, one can use ‘dimensional regularization’ approach introduced in the previous section. In this scheme we merely set $\delta = 0$ and keep n in eq. (B.33) unspecified. As a result, the power law divergences are eliminated and we find

$$\begin{aligned}
S_{\alpha, \text{finite}}^{(1/2)} &= \frac{(1+\alpha)}{\alpha} \frac{\mathcal{A}_\Sigma}{48(2\pi)^n \sqrt{\pi}} \sum_{j=0}^n \frac{a_{j,n}^{(1/2)}}{R^{2j}} \int_0^\infty dt t^{j-n-\frac{3}{2}} e^{-tm^2} \\
&= \frac{(1+\alpha)}{\alpha} \frac{\mathcal{A}_\Sigma}{48(2\pi)^n \sqrt{\pi}} m^{2n+1} \sum_{j=0}^n \frac{a_{j,n}^{(1/2)}}{(mR)^{2j}} \Gamma(j-n-1/2) \\
&= \frac{1+\alpha}{24\alpha} \frac{(-1)^{\frac{D-1}{2}} \pi}{(2\pi)^{\frac{D-2}{2}} \Gamma(D/2) \sqrt{2}} \mathcal{A}_\Sigma \left(m^{D-2} - a_{1, \frac{D-3}{2}}^{(1/2)} \frac{D-2}{2} \frac{m^{D-4}}{R^2} + \dots \right),
\end{aligned} \tag{B.35}$$

where we have simplified the final expression with eq. (B.14). Note that we are expanding the final result in the limit $mR \gg 1$. The leading contribution in this limit precisely matches the expected area term (1.3) with the pre-factor given by eq. (1.6).

The sub-leading term above also agrees with eqs. (2.57) and (2.66) for $D = 3$ and 5 , respectively, if we replace $R^2 \rightarrow -R^2$ and substitute $a_{1,0}^{(1/2)} = 0$ and $a_{1,1}^{(1/2)} = 1/2$ using eq. (B.31). We might remark that, in fact, the ζ -function method can also be applied here to derive the same result. In particular, this calculation confirms the result $a_{1, \frac{D-3}{2}}^{(1/2)}$ given in eq. (B.32). Following our discussion for the scalar fields, we can use this general result to identify the form of this subleading term for arbitrary D . We can then re-express this contribution in the covariant form:

$$S_{\text{univ}} = \frac{D-2}{24} \gamma_{D, \text{fermion}} \int_\Sigma d^{D-2} \sigma \sqrt{h} \mathcal{R}(h) m^{D-4}. \tag{B.36}$$

Again $\mathcal{R}(h)$ is the Ricci scalar of the metric induced on the entangling surface while the coefficient $\gamma_{D, \text{fermion}}$ is given in eq. (1.6). This expression is only applicable for odd $D \geq 5$.

Even dimensions

The computation of the Rényi entropy for even dimensions is, of course, similar to that for odd dimensions. However, a systematic evaluation of all the divergences using

the simple regularization of the heat kernel (2.8) requires more effort for even d . The extra computational complications originate from the fact that heat kernel of the Dirac operator on \mathbb{H}^2 cannot be expressed in terms of elementary functions. Instead we have the integral representation in eq. (B.24).

To demonstrate the procedure, we start from the special case $d = 2$ and evaluate all the divergences in this case. The extension of this computation to higher even d presents no conceptual difficulties. Therefore rather than pursuing this approach (2.8) to construct a voluminous general expression which contains all of the (nonuniversal) power law divergences for general even d , we shift our focus to only evaluating the universal logarithmic contributions. In this case, the ζ -function approach (2.17) provides the most efficient method to produce a general result.

$d = 2$ ($D = 4$):

In this case we need eq. (B.23) with $n = 0$. In particular then, the limit of coincident points gives

$$K_{\mathbb{H}^2}^{(1/2)}(t, x, x) = f_{\mathbb{H}^2}^{(1/2)}(0, t) \mathbb{I}_2 . \quad (\text{B.37})$$

Again the two-by-two identity matrix \mathbb{I}_2 corresponds the coincident point limit of the spinor propagator $\hat{U}(x, y)$ on \mathbb{H}^2 . Given the expression in eq. (B.24), we evaluate $f_{\mathbb{H}^2}^{(1/2)}(0, t)$ by expanding for small t

$$\begin{aligned} f_{\mathbb{H}^2}^{(1/2)}(0, t) &= \frac{1}{(4\pi)^{\frac{3}{2}} R \sqrt{t}} \int_0^\infty x e^{-x^2/4} \coth\left(\frac{\sqrt{t} x}{2R}\right) dx \\ &= \frac{1}{4\pi t} \left(1 + \frac{t}{6R^2} + \dots\right) . \end{aligned} \quad (\text{B.38})$$

Here we only explicitly show the terms which contribute to the divergences in the Rényi entropy and the ellipsis denotes terms which only make finite contributions. Proceeding as usual with this result, we find

$$\begin{aligned} S_\alpha^{(1/2)} &= \frac{(1 + \alpha)}{\alpha} \frac{\text{Vol}(\mathbb{H}^2)}{48\pi} \int_{\delta^2}^\infty \frac{dt}{t^2} e^{-tm^2} \left(1 + \frac{t}{6R^2} + \dots\right) , \\ &= \frac{1 + \alpha}{48\pi\alpha} \mathcal{A}_\Sigma \left(\frac{1}{\delta^2} + \left(2m^2 - \frac{1}{3R^2}\right) \log(m\delta) + \dots \right) \end{aligned} \quad (\text{B.39})$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{H}^2)$. Of course, the leading divergence above corresponds to the usual ‘area law’ term. For $\alpha = 1$, the second term precisely matches eq. (1.3) with pre-factor given by eq. (1.6) with $D = 4$. Finally, eq. (B.39) agrees with the analogous result

(2.62) for a spherical waveguide after replacing $R^2 \rightarrow -R^2$.

General even $d \geq 4$

Here we use the spinor ζ -function (B.27) to compute the partition function (2.17) and then the Rényi entropy (2.4). Evaluating eq. (B.27) at $z = 0$ and substituting the result into eq. (2.17) leads to the following universal contribution to the Rényi entropy

$$S_{\alpha, \text{univ}}^{(1/2)} = -\frac{(1+\alpha)}{\alpha} \frac{\text{Vol}(\mathbb{H}^d)}{12(2\pi)^{d/2}\Gamma(d/2)R^d} \times \sum_{k=0}^{(d-2)/2} h_{k,d}^{(1/2)} \left[4 \int_0^\infty \frac{x^{2k+1}}{e^{2\pi x} - 1} dx + \frac{(-R^2 m^2)^{k+1}}{k+1} \right] \log(m\delta) . \quad (\text{B.40})$$

Substituting $d = 2$, we recover the universal part appearing in eq. (B.39) above. Substituting in other explicit values of d , one can generate universal contributions for higher even dimensions. Thus, for instance, we obtain with $d = 4$ ($D = 6$)

$$S_{\alpha, \text{univ}}^{(1/2)}(d = 4) = -\frac{(1+\alpha)}{96\pi^2\alpha} \text{Vol}(\mathbb{H}^4) \left(m^4 - 2\frac{m^2}{R^2} + \frac{11}{30} \frac{1}{R^4} \right) \log(m\delta) . \quad (\text{B.41})$$

We can also expand the above expression in the limit $mR \gg 1$ to determine a general expression for the leading terms:

$$S_{\alpha, \text{univ}}^{(1/2)} = \frac{(1+\alpha)}{12\alpha} \frac{(-)^{D/2} \mathcal{A}_\Sigma}{(2\pi)^{\frac{D-2}{2}} \Gamma(D/2)} \left(m^{D-2} - \frac{(D-2)^2(D-3)}{24} \frac{m^{D-4}}{R^2} + \dots \right) \log(m\delta) , \quad (\text{B.42})$$

where $\mathcal{A}_\Sigma = \text{Vol}(\mathbb{H}^{D-2})$. The leading term has the expected form (1.3) with the pre-factor given (1.6). The next-to-leading term reveals a new universal curvature contribution (1.5). Let us turn to the entanglement entropy by setting $\alpha = 1$ and then write this subleading contributions in a covariant form as

$$S_{\text{univ}} = \frac{D-2}{24} \gamma_{D, \text{fermion}} \int_\Sigma d^{D-2}\sigma \sqrt{h} \mathcal{R}(h) m^{D-4} \log(m\delta) . \quad (\text{B.43})$$

Again $\mathcal{R}(h)$ is the Ricci scalar of the metric induced on the entangling surface and the coefficient $\gamma_{D, \text{fermion}}$ is given in eq. (1.6). Of course, this contribution only appears for even $D \geq 4$.

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